Ideals Generated by Principal Minors
Joint Mathematics Meetings 2014

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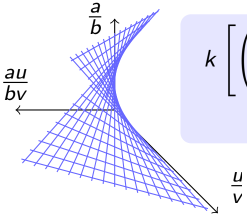


$$k\left[\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right] \Big/ (x_{11}x_{22} - x_{12}x_{21})$$
$$\cong k[au, bu, av, bv] \subset k[a, b, u, v]$$

(affine open set of a) quadric ruled surface in $\mathbb{P}^3$

Thank-you for the invitation to speak!

Given a matrix of indeterminates, we can form a polynomial ring over an arbitrary algebraically closed field, k.

r, s arbitrary positive integers

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1s} \\ \vdots & \ddots & \ddots & \vdots \\ x_{r1} & \cdots & \cdots & x_{rs} \end{pmatrix}$$

$k[X] =$ polynomial ring over k with variables $x_{11}, \ldots, x_{rs}$

$G = \mathrm{GL}_t\, k$, general linear group

(equivalently, invertible $t \times t$ matrices over k)

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1t} \\ \vdots & \ddots & \ddots & \vdots \\ y_{r1} & \cdots & \cdots & y_{rt} \end{pmatrix}; \qquad Z = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1s} \\ \vdots & \ddots & \ddots & \vdots \\ z_{t1} & \cdots & \cdots & z_{ts} \end{pmatrix}$$

S = polynomial ring over k, in the variables $y_{11}, \ldots, y_{rt}, z_{11}, \ldots, z_{ts}$

S^G = k-algebra generated by the entries of the product matrix YZ

Let (each matrix $M \in$) G act on S by matrix multiplication:

$$M: \quad Y \mapsto YM^{-1} \qquad Z \mapsto MZ$$

$y_{ij} \mapsto (i, j)$ entry of YM^{-1}
$(0 \leq i \leq r,\ 0 \leq j \leq t)$

$z_{ij} \mapsto (i, j)$ entry of MG
$(0 \leq i \leq t,\ 0 \leq j \leq s)$

The k-map

$$\Phi : k[X] \to S$$

$$x_{ij} \mapsto (i, j) \text{ entry of } YZ$$

$$(1 \le i \le r,\ 1 \le j \le s)$$

$S^G = \{s \in S \mid g(s) = s \text{ for all } g \in G\}$ is called the **ring of invariants** under the action of G.

$\mathrm{I}_{t+1} = \ker \Phi$ is called a **determinantal ideal.**

induces a surjection $k[X] \twoheadrightarrow S^G \subset S$, and $\ker \Phi$ is generated by the size $t+1$ minors of X. To maintain consistency with convention, put $t' = t+1$. $k[X]/\mathrm{I}_{t'} \cong S^G$ is called a **determinantal ring**.

Theorem (Eagon, Hochster 1971)

Determinantal rings are Cohen-Macaulay.

$r \leq s$
$G = \mathrm{SL}_r\, k$, special linear group
(i.e., matrices with determinant equal to 1)

G acts on $k[X]$ via

$$M : x_{ij} \mapsto (i,j) \text{ entry of } MX \quad (1 \leq i \leq r,\ 1 \leq j \leq s).$$

$$\det M = \wedge^r M \implies \wedge^r(MX) = \wedge^r X$$
$$\implies r\text{-minors of } X \text{ remain fixed}$$

The ring of invariants is exactly the homogeneous coordinate ring of the **Grassmann variety** (we use $\mathrm{Grass}_k(r,s)$ to denote said variety). The relations on its k-generators (the r-minors) are exactly the **Plücker relations**.

X is an $n \times n$ alternating matrix. The **Pfaffian ideals** $\mathrm{Pf}_t = \mathrm{Pf}_t(X)$ are generated by the square roots of each of the symmetrically placed t-minors of X.

$$X = \begin{pmatrix} 0 & x_1 & x_2 & x_3 & \cdots & x_{n-1} \\ -x_1 & 0 & x_n & x_{n+1} & \cdots & x_{2n-3} \\ -x_2 & -x_n & 0 & x_{2n-2} & & \\ -x_3 & -x_{n+1} & -x_{2n-2} & 0 & & \\ \vdots & & & & & \\ -x_{n-1} & x_{2n-3} & & & & \end{pmatrix}$$

$$k[X] = k[x_1, \ldots, x_{\binom{n}{2}}]$$

$$\mathrm{Pf}_2 = (x_1, \cdots, x_{\binom{n}{2}})$$
$$\mathrm{Pf}_3 = (0)$$
$$\mathrm{Pf}_4 = (x_1 x_{2n-2} + x_2 x_{n+1} + x_3 x_n, \ldots)$$
$$\vdots$$
$$\mathrm{Pf}_n = \left(\begin{cases} (\sqrt{\det X} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \right)$$

The **Pfaffian rings** are the quotients $k[X]/\mathrm{Pf}_t$.

Ideals Generated by Principal Minors

The **principal** minors of a square matrix are those whose defining row and column indices are the same.

Question

What geometric properties do algebraic sets defined by principal minors satisfy?

$\mathfrak{P}_t =$ ideal in $k[X]$ generated by the size t principal minors of the generic square matrix X

Theorem (–)

For all n, $k[X]/\mathfrak{P}_2$ is a normal complete intersection domain, and is isomorphic to a k-algebra generated by monomials. Hence, it is strongly F-regular and Gorenstein.

For $t > 2$ it becomes more convenient to study components of $\mathcal{V}(\mathfrak{P}_t)$ according to matrix rank.

$$\mathcal{Y}_{n,r,t} = \mathcal{V}(\mathfrak{P}_t) \bigcap \{n \times n \text{ matrices of rank } r\}$$

Theorem (–)

For $n \geq 4$, $\mathcal{V}(\mathfrak{P}_{n-1})$ has two components: one of codimension 4 and the other of codimension n.

(1) $$\mathcal{V}(\mathrm{I}_{n-1}) = \bigcup_{r' < n-1} \mathcal{Y}_{n,r',n-1}$$

(2) $$\mathcal{V}(\mathfrak{Q}) = \overline{\mathcal{Y}}_{n,n,n-1} \supset \mathcal{Y}_{n,n-1,n-1}$$

$$\Theta : k[X]_{\det X} \to \left(\frac{k[X]}{\mathfrak{P}_1}\right)_{\det X}$$

$$X \mapsto (\det X) \cdot X^{-1}$$

$$\mathfrak{Q} = \text{contraction of } \ker \Theta \text{ to } k[X]$$

For any n, t:

$$\begin{pmatrix} x_{11} & \cdots & x_{1,t+1} & x_{1,t+2} & \cdots & \cdots & x_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ x_{t+1,1} & \cdots & x_{t+1,t+1} & x_{t+1,t+2} & \cdots & \cdots & x_{t+1,n} \\ 0 & \cdots & \cdots & 0 & x_{t+2,t+2} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

$$\begin{aligned} \operatorname{ht} \mathfrak{P}_t &\leq n + n - 1 + n - 2 + \cdots + n - (n - t - 2) + 4 \\ &= \binom{n+1}{2} - \binom{t+2}{2} + 4. \end{aligned}$$

Ideals Generated by Principal Minors

$t = n - 2$

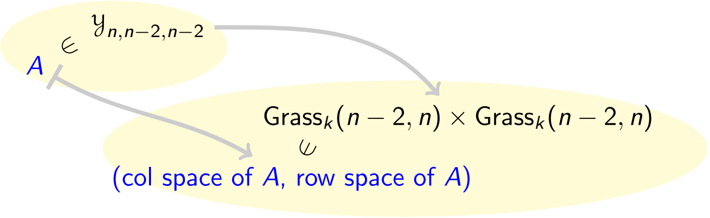
Theorem (–)

$$\dim \mathcal{Y}_{n,n-2,n-2} = n^2 - 4 - n$$

In explaining this, we will focus on the first non-trivial case, $n = 5$. A matrix $A \in \mathcal{Y}_{5,3,3}$ iff

(a) rank $A = 3$ and

(b) the size 3 principal minors of A vanish.



Wolog, say the minor given by row indices $I = \{1,2,3\}$ and column indices $J = \{1,2,4\}$ does not vanish.

$2(n-2) + (n-2)^2 + 2(n-2) = \mathbf{n^2 - 4}$ parameters

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ u_{41} & u_{42} & u_{43} \\ u_{51} & u_{52} & u_{53} \end{pmatrix}}_{U} \cdot W \cdot \underbrace{\begin{pmatrix} 1 & 0 & v_{13} & 0 & v_{15} \\ 0 & 1 & v_{23} & 0 & v_{25} \\ 0 & 0 & v_{33} & 1 & v_{35} \end{pmatrix}}_{V}$$

W: 3×3 invertible

The principal 3 minors of A vanish

$\iff$ the diagonal of $\wedge^3 A$ vanishes

$\iff$ for each index $i = 1, \ldots, 10$ the ith entry of either the column vector $\wedge^3 U$ or the row vector $\wedge^3 V$ vanishes.

$$\wedge^3 A = \overbrace{\begin{pmatrix} 1 \\ u_{43} \\ u_{53} \\ -u_{42} \\ -u_{52} \\ u_{42}u_{53} - u_{43}u_{52} \\ u_{41} \\ u_{51} \\ -u_{41}u_{53} + u_{43}u_{51} \\ -u_{41}u_{52} + u_{42}u_{51} \end{pmatrix}}^{\wedge^3 U} \cdot \wedge^3 \underset{\substack{\nwarrow \\ \text{scalar}}}{W}$$

$$\cdot \underbrace{\begin{pmatrix} v_{33} & 1 & v_{35} & v_{23} & v_{23}v_{35} - v_{25}v_{33} & -v_{25} & -v_{13} & -v_{13}v_{35} + v_{15}v_{33} & v_{15} & v_{13}v_{25} - v_{15}v_{23} \end{pmatrix}}_{\wedge^3 V}$$

The pair

$$\mathbf{g}_U \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ u_{41} & u_{42} & u_{43} \\ u_{51} & u_{52} & u_{53} \end{pmatrix} \in \text{Grass}_k(3,5) \times \text{Grass}_k(3,5) \ni \mathbf{g}_V \sim \begin{pmatrix} 1 & 0 & v_{13} & 0 & v_{15} \\ 0 & 1 & v_{23} & 0 & v_{25} \\ 0 & 0 & v_{33} & 1 & v_{35} \end{pmatrix}$$

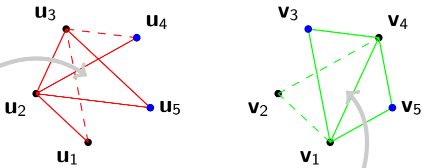
yields, in order for the diagonal of the product to vanish (up to a scalar), the system

$$\left.\begin{array}{r} 1(v_{33}) \\ u_{43}(1) \\ u_{53}v_{35} \\ -u_{42}v_{23} \\ -u_{52}(v_{23}v_{35} - v_{25}v_{33}) \\ (u_{42}u_{53} - u_{43}u_{52})(-v_{25}) \\ u_{41}(-v_{13}) \\ u_{51}(-v_{13}v_{35} + v_{15}v_{33}) \\ (-u_{41}u_{53} + u_{43}u_{51})v_{15} \\ (-u_{41}u_{52} + u_{42}u_{51})(v_{13}v_{25} - v_{15}v_{23}) \end{array}\right\} = 0.$$

Graphs of Plücker Coordinates
Encoding Vanishing Minors

For any point $\mathbf{g} \in \text{Grass}_k(3,5)$, we encode which of its Plücker coordinates vanish into a simple graph with 5 vertices. A vertex represents an index; an edge joining two vertices indicates the minor of complementary indices vanishes.

A **simple** graph has no loops and for every pair of vertices, has at most one edge joining them.



$$\mathbf{g}_U \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ u_{41} & u_{42} & u_{43} \\ u_{51} & u_{52} & u_{53} \end{pmatrix} \qquad \mathbf{g}_V \sim \begin{pmatrix} 1 & 0 & v_{13} & 0 & v_{15} \\ 0 & 1 & v_{23} & 0 & v_{25} \\ 0 & 0 & v_{33} & 1 & v_{35} \end{pmatrix}$$

U, V are normalized $\implies$ Plücker coordinates for $\mathbf{g}_U$, $\mathbf{g}_V$ identify with minors of the submatrices consisting of the variables

Proposition

The graph determined by $\mathbf{g} \in \mathrm{Grass}_k(n-2,n)$ *is well-defined.*

Definition

1. A size n graph G is **permissible** means
   (a) G has at most $\binom{n}{n-2} - 1$ edges, i.e., G is not complete and
   (b) every vertex in G of degree $a < n-1$ is part of a complete subgraph of size a (called a **clique**).
2. A subvariety $\mathcal{S} \subseteq \mathrm{Grass}(n-2,n)$ is **permissible** means it is the set of all points with the same permissible graph, which we denote $\mathrm{Graph}(\mathcal{S})$.

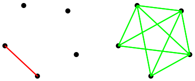
Graphs of Plücker Coordinates

Examples ($n = 5$)

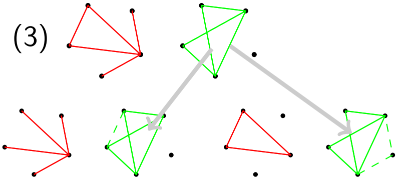
A vertex is

1. **isolated** means it has no edges.
2. **dominating** means it is joined to every other vertex.

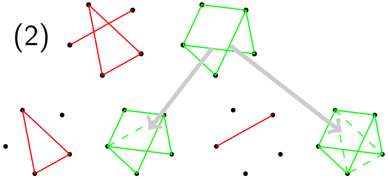
(1)



(3)



(2)



A finite set of graphs $G_1, G_2, \ldots, G_s$ is a **graph covering** means the union of their edges forms a complete graph on a set of vertices $1, \ldots, n$.

Question

What are the minimal pairs of permissible graphs that cover n vertices?

Characterization of the Components of $\mathcal{Y}_{n,n-2,n-2}$

Theorem (–)

The pairs $\mathcal{S}, \mathcal{T} \subset \operatorname{Grass}(n-2, n)$ *corresponding to components of* $\mathcal{Y}_{n,n-2,n-2}$ *are those such that*

(a) $\operatorname{Graph}(\mathcal{S})$ *consists of a size* $a > 1$ *clique with* $n - a$ *isolated vertices and*

(b) $\operatorname{Graph}(\mathcal{T})$ *is the complement of* $\operatorname{Graph}(\mathcal{S})$, *i.e., a size* n *graph with* $n - a$ *dominating vertices.*

Theorem (–)

Suppose $\mathcal{S} \times \mathcal{T} \subseteq \operatorname{Grass}(n-2, n) \times \operatorname{Grass}(n-2, n)$ *corresponds to a component of* $\mathcal{Y}_{n,n-2,n-2}$. *Let* a *denote the size of the maximal clique occurring in* $\operatorname{Graph}(\mathcal{S})$. *Then*

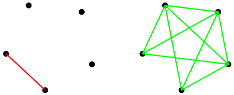
1. $\operatorname{codim} \mathcal{S} = a - 1$.
2. $\operatorname{codim} \mathcal{T} = 2(n - a)$.
3. *(corollary)* $2 \leq a \leq n - 1$, *so the minimal codimension of such* $\mathcal{S} \times \mathcal{T}$ *is* n.

Characterization of the Components of $\mathcal{Y}_{n,n-2,n-2}$

$n = 5$

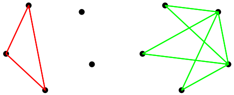
(Graph($\mathcal{S}$) Graph($\mathcal{T}$))

(1)



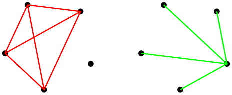
codim($\mathcal{S} \times \mathcal{T}$) $= 7$

(2)



codim($\mathcal{S} \times \mathcal{T}$) $= 6$

(3)



codim($\mathcal{S} \times \mathcal{T}$) $= 5$

(up to, of course, permutation of $\mathcal{S}$ and $\mathcal{T}$)

What's Next

Question

1. What are the components for $\mathcal{Y}_{n,n-1,n-2}$ and $\mathcal{Y}_{n,n,n-2}$?
2. What are the components for $\mathcal{Y}_{n,n-3,n-3}$?
3. What are the singularities of $\mathcal{V}(\mathfrak{Q})$ like?