# Finiteness of associated primes of local cohomology modules over Stanley-Reisner rings

joint w/ R. Barrera and J. Madsen

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## Thank-you for the invitation to speak!

**Local cohomology** has applications to Cosmology and String Theory, and it is one of the most active research areas in Commutative Algebra.

Little is known about local cohomology modules.

# Local cohomology modules

- $R =$ commutative Noetherian ring with $1$
- $I =$ ideal in $R$
- $M$=$R$-module (may or may not be Noetherian or finitely generated)
- $j =$ non-negative integer

The $j$**th local cohomology module of** $M$ **with support in** $I$ is defined as the following direct limit of $\mathrm{Ext}$ modules:

$$H_I^j(M) = \varinjlim_t \mathrm{Ext}_R^j(R/I^t, M).$$

It is the right derived functor of $H^0_I(?)$:

$$\begin{aligned} H^0_I(M) :=& \bigcup_t \operatorname{Ann}_M I^t \\ =& \{u \in M \mid uI^t = 0 \text{ for some } t\} \\ =& \varinjlim_t \operatorname{Hom}_R(R/I^t, M) (= \varinjlim_t \operatorname{Ext}^0_R(R/I^t, M)) \end{aligned}$$

the global sections of the sheaf $\tilde{M}$ with support on the closed subscheme $\operatorname{Spec} R/I \subset \operatorname{Spec} R$.

$H^1_I(M)$ measures the obstruction to extending a section of a sheaf to a global section; put $\mathcal{X} = \operatorname{Spec} R$ and $\mathcal{U} = \mathcal{X} \setminus \operatorname{Spec}(R/I)$

$$0 \to H^0_I(M) \to H^0(\mathcal{X}, \tilde{M}) \to H^0(\mathcal{U}, \tilde{M}|_{\mathcal{U}}) \to H^1_I(M) \to 0$$

If $(R, \mathfrak{m})$ is a local ring and $M$ is finite generated, then $H^j_{\mathfrak{m}}(M)$ can detect regular sequences, compute depth, and reveal the Cohen-Macaulay and Gorenstein properties.

In practice, $H_I^j(M)$ is the $j$th cohomology module of the **Čech complex**

$$0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i<j} M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_s} \to 0$$

where:

- $f_1, \ldots, f_s \in R$ generate $I$ up to radical
- given any $f \in R$ and any $R$-module $N$, $N_f = N \otimes_R R_f$, and $R_f = R[\frac{1}{f}]$ is the **localization of $R$ at $f$**
- the maps in the complex are the natural localization maps $u \mapsto \frac{u}{1}$

# Localization

Localizing at a prime ideal gives the **stalk at a point** in the **Zariski topolgy**:

$$\begin{aligned}\operatorname{Spec} R &= \{\text{prime ideals in } R\} \leftarrow \text{ topological space}\\ \mathcal{V}(J) &= \{\text{primes containing the ideal } J\} \leftarrow \text{ its closed sets}\\ &= \operatorname{Spec} R/J\end{aligned}$$

$R$ **localized at** $P$ is given by $R_P = R\left[\frac{1}{f} \mid f \in R \setminus P\right]$ and $N_P = N \otimes_R R_P$

Localization is **flat**; as a result, many questions can be reduced to the local case (**local-global principle**).

(statement about an $R$-module $N$ is true)

$$\Leftrightarrow$$

(same statement about $N_P$ is true for all $P \in \operatorname{Ass}_R N$)

- $\operatorname{Ass}_R N$ = **assassinator of $N$**, set of all primes associated to $N$
- **$P$ is associated to $N$** means $P = \operatorname{Ann}_R(u)$, the set of ring elements that annihilate some element $u \in N$; equivalently, $P \in \operatorname{Ass}_R N$ if and only if $R/P$ is isomorphic to a submodule of $N$.

Local cohomology is the local-global analogue to **sheaf cohomology**.

# Finiteness of associated primes

Our project is motivated by the following:

**Question (C. Huneke 1990)**

Do the local cohomology modules over a Noetherian ring $R$ have finitely many associated primes?

(Answer: No.)

# Counterexamples

- A. Singh 2000: $R = \dfrac{\mathbb{Z}[u,v,w,x,y,z]}{(ux+vy+wz)} \quad \Longrightarrow \quad |\operatorname{Ass}_R(H^3_{(x,y,z)}R)| = \infty$

  Reason: This local cohomology module has $p$-torsion for all primes $p \in \mathbb{Z}$.

- M. Katzman 2002:

$$R = \frac{K[s,t,u,v,x,y]}{(su^2x^2 - (s+t)uvxy + tv^2y^2)} \quad (K = \text{ any field})$$

$$\Longrightarrow \quad |\operatorname{Ass}_R(H^2_{(x,y)}R)| = \infty$$

Also shows torsion for infinitely many ring elements. Unlike in Singh's example, this ring is local.

- Singh and I. Swanson 2004: generalized Katzman's results with examples of normal hypersurfaces
  - of characteristic $0$ with rational singularities and
  - of characteristic $p$ that are $F$-regular

Are there instances where the answer is yes? (Yes.)

# Affirmatives

- M. Hellus 2001: $M = R$ is Cohen-Macaulay and
  - $\operatorname{Ass}_R(H^3_{(x,y)}R)$ is finite for every $x, y \in R$
  - $\operatorname{Ass}_R(H^3_{(x_1,x_2,y)}R)$ is finite for $x_1, x_2 \in R$ a regular sequence and $y \in R$
- T. Marley 2001: $(R, \mathfrak{m})$ is local, $M$ is finitely generated and
  - $\dim R \leq 3$
  - $\dim R = 4$ and $R$ is regular on the punctured spectrum ($\operatorname{Spec} R \setminus \mathfrak{m}$ is smooth)
  - $\dim R = 5$, $R$ is unramified, regular, and $M$ is torsion-free
- Marley and J. Vassilev 2002: $M$ is finitely generated and
  - $\dim M \leq 3$
  - $\dim R \leq 4$
  - $\dim M/IM \leq 2$ and $M$ satisfies Serre's condition $S_{\dim M-3}$
  - $\dim M/IM \leq 3$, $\operatorname{Ann}_R M = 0$, $R$ is unramified, and $M$ satisfies $S_{\dim M-3}$

- S. Takagi and R. Takahashi 2008: $M = \omega_R$, the canonical module of a Cohen-Macaulay ring of finite $F$-representation type (FFRT) $\implies$ affirmative for $M = R$ Gorenstein of FFRT
- H. Robbins 2014: $M = R$ is a polynomial or power series ring over a two- or three- dimensional normal domain with an isolated singularity, finitely generated over a field of characteristic $0$
- B. Bhatt, M. Blicklé, G. Lyubeznik, Singh, and W. Zhang (BBLSZ) 2014: $M = R$ is a smooth $\mathbb{Z}$-algebra
  Idea: In the smooth case, $p$-torsion can be controlled.

Proving anything more broad has been HARD!

# Regular in characteristic $p$

**Theorem (Huneke and R. Sharp 1993)**

*Yes, when $R$ is a regular ring containing a field of characteristic $p > 0$.*

Regular rings are pretty nice...
(regular $\implies$ complete intersection $\implies$ Gorenstein $\implies$ Cohen-Macaulay)
but it's still a fairly broad class of rings.

Why characteristic $p$?

To prove things in characteristic $p$: use the **Frobenius map**

$$\begin{aligned} F^e : R &\to R \\ r &\mapsto r^{p^e}. \end{aligned}$$

- In characteristic $p$ it's a ring map! $(F^e(r+s) = (r+s)^{p^e} = r^{p^e} + s^{p^e} = F^e(r) + F^e(s))$
- $F^e(R) \cong R$ as rings
- When $R$ is regular, it is locally free as a module over itself.

Huneke & Sharp: It suffices to show for $R$ local, and so write $R \cong F^e(R)^{\oplus m_e} (\cong R^{\oplus m_e})$ as modules.

Then using the Ext definition of local cohomology:

$$\begin{aligned}
\operatorname{Ass}_R(H^j_I M) &= \operatorname{Ass}_{F^e(R)}\left((H^j_I M)\otimes_R F^e(R)\right)\\
&= \operatorname{Ass}_{F^e(R)}\left(\varinjlim_t \operatorname{Ext}^j_R(R/I^t, M)\otimes_R F^e(R)\right)\\
&= \operatorname{Ass}_{F^e(R)}\left(\varinjlim_e \operatorname{Ext}^j_R(R/I^{p^e}, M)\otimes_R F^e(R)\right)\\
&= \operatorname{Ass}_{F^e(R)}\left(\varinjlim_e \operatorname{Ext}^j_{F^e(R)}(R/I^{p^e}\otimes_R F^e(R), M\otimes_R F^e(R))\right)\\
&= \operatorname{Ass}_{F^e(R)}\left(\varinjlim_e \operatorname{Ext}^j_{F^e(R)}(F^e(R)/IF^e(R), M\otimes_R F^e(R))\right)\\
&= \operatorname{Ass}_R\left(\varinjlim_e \operatorname{Ext}^j_R(R/I, M\otimes_R R\right)\\
&= \operatorname{Ass}_R\left(\varinjlim_e \operatorname{Ext}^j_R((R/I)^{\oplus m_e}, M\right)\\
&= \operatorname{Ass}_R\left(\varinjlim_e \operatorname{Ext}^j_R(R/I, M)^{\oplus m_e}\right)\\
&\subseteq \operatorname{Ass}_R\left(\operatorname{Ext}^j_R(R/I, M)\right)
\end{aligned}$$

# Regular local in characteristic $0$

**Theorem (Lyubeznik 1993)**

*Yes, it is also true for a regular local ring containing a field of characteristic $0$.*

Uses the burgeoning theory of $\mathcal{D}$-modules.

Actually proved a stronger result: *For $R$ regular containing a field of characteristic $0$, and any maximal ideal $\mathfrak{m}$ in $R$, the number of associated primes of $H_I^j(M)$ contained in $\mathfrak{m}$ is finite.*

J.-E. Björk 1979: Over a formal power series ring in finitely many variables over a field of characteristic $0$, there exists a class of **holonomic** $\mathcal{D}$-modules, to which local cohomology modules belong.

An associated prime in $\mathfrak{m}$ is the restriction of a prime in the completion $\hat{R}$ of $R$ with respect to $\mathfrak{m}$, that is associated to $H^j_{I\hat{R}}(M \otimes_R \hat{R}) \cong H^j_I(M) \otimes_R \hat{R}$.

Cohen's Structure Theorem: $\hat{R} \cong R/\mathfrak{m}[[x_1, \ldots, x_n]]$
$\implies H^j_I(M) \otimes_R \hat{R} = H^j_{\hat{I}}(\hat{M})$ is holonomic.

Holonomic modules are semisimple $\implies$ finitely many associated primes.

# Equicharacteristic

Later, Lyubeznik somewhat reconciled the characteristic $p$ and $0$ cases.

**Theorem (Lyubeznik 2000)**

*For $R$ regular containing a field of characteristic $p$ or $0$, and any maximal ideal $\mathfrak{m}$ in $R$, the number of associated primes of $H_I^j(M)$ contained in $\mathfrak{m}$ is finite.*

The proof reduces to one or the other characteristic, after which the techniques are different.

Lyubeznik: If $A$ is any regular ring containing a field and for all $f \in A$, the localized rings $A_f$ have finite $\mathcal{D}$-length, then the local cohomology modules over $A$ all have finitely many associated primes.

Lyubeznik 1997: Over [a finitely generated algebra over] a formal power series ring $A$ in finitely many variables over a field of characteristic $p$, there exists a class of $F$-**finite** $F$-modules, to which local cohomology modules belong. So do the localizations $A_f$.

An associated prime in $\mathfrak{m}$ is the restriction of a prime in the completion $\hat{R}$ of $R$ with respect to $\mathfrak{m}$, that is associated to $H^j_{I\hat{R}}(M \otimes_R \hat{R}) \cong H^j_I(M) \otimes_R \hat{R}$.

Cohen's Structure Theorem: $\hat{R} \cong R/\mathfrak{m}[[x_1, \ldots, x_n]]$.

Björk 1979: The localizations $\hat{R}_f$ have finite $\mathcal{D}$-length in characteristic $0$.

Lyubeznik 1997: The localizations $\hat{R}_f$ are $F$-finite in characteristic $p$;

$F$-finite $\implies$ finite $F$-length.

finite $F$-length $\implies$ finite $\mathcal{D}$-length.

# Mixed characteristic

**Theorem (Lyubeznik 2000)**

*Also yes, when $R$ is regular local and unramified.*

Proof reduces to the known results in equicharacteristic.

# Characteristic-free

**Theorem (Lyubeznik 2010)**

*Affirmative when $R$ is a polynomial ring in finitely many variables over a field.*

This was already known, as a consequence of Björk's results. But this is the first truly characteristic-free proof.

Key ingredient: Updated notion of holonomicity by V. Bavula (2009).

# Our main result

We use methods very similar to Lyubeznik to show the following:

**Theorem (BMW 2015)**

*If $R$ is a Stanley-Reisner ring over a field and its associated simplicial complex is a $T$-space, then the set of associated primes of any local cohomology module over $R$ is finite.*

# Stanley-Reisner rings

- $S = K[x_1, \ldots, x_n]$, the polynomial ring over a field $K$
- $\Delta =$ simplicial complex with vertices labelled by the variables $x_1, \ldots, x_n$
- $I_\Delta = (x_{i_1} \cdots x_{i_t} \mid \{x_{i_1}, \ldots, x_{i_t}\} \notin \Delta)S$ is called the **face ideal of $\Delta$ over $K$**

$K[\Delta] = S/I_\Delta$ is called the **Stanley-Reisner ring of $\Delta$ over $K$**.

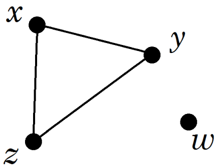
To each face $F \in \Delta$ corresponds a prime ideal $P_F$ generated by the variables not appearing in $F$.

In fact, the minimal primes (minimal in $\operatorname{Ass}_R R/(0)$ with respect to containment) are in bijection with the facets of $\Delta$.

# $T$-spaces

A simplicial complex $\Delta$ is a **$T$-space** means for every face $F \in \Delta$, if $x \notin F$ then there exists a facet in $\Delta$ containing $F$ but not $x$.

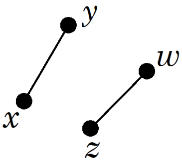
## Example




$$\Delta = \{\{x,y\},\{x,z\},\{y,z\},\{w\},\{x\},\{y\},\{z\}\}$$
$$I_\Delta = (xw, yw, zw, xyz, xyw, xzw, yzw)S$$
$$R = S/I_\Delta = \frac{K[x,y,z,w]}{(z,w)\cap(y,w)\cap(x,w)\cap(x,y,z)}$$

Is (the simplicial complex associated to) $R$ a $T$-space? (Yes.)

## Example




$$\Delta = \{\{x,y\},\{z,w\},\{x\},\{y\},\{z\},\{w\}\}$$
$$I_\Delta = (xz, xw, yz, yw, xyz, xyw, xzw, yzw)S$$
$$R = S/I_\Delta = \frac{K[x,y,z,w]}{(z,w)\cap(x,y)}$$

Is it a $T$-space? (No.) In fact, a graph is a $T$-space if and only if none of its vertices have degree $1$.

# Finite length

Consensus says the problem with studying local cohomology modules is that they are just too big. We want ways to "control" their size.

Common approach: Construct a **filtration** of $R$-submodules

$$0 = N_0 \subset N_1 \subset \cdots \subset N_l = N$$

in such a way that each of the factors $N_i/N_{i-1}$ has finitely many associated primes.

We get the result by the containment

$$\mathrm{Ass}_R(N) \subset \bigcup_i \mathrm{Ass}_R(N_i/N_{i-1}),$$

provided the filtration has finite length.

Problem: When $N$ is not finitely generated (e.g., $N = H_I^j M$) it is HARD! to prove it has finite length.

Strategy: Show finite length over a larger ring, an $R$-algebra. For example, $\mathcal{D}$.

Lyubeznik 2000: To show local cohomology modules have finite $\mathcal{D}$-length it is enough to show $R_f$, for any $f \in R$, has finite $\mathcal{D}$-length.

- consequence of the Čech complex definition of local cohomology
- Proving $R_f$ has finite $\mathcal{D}$-length is still HARD! (recall Björk's result from earlier)

# Rings of differential operators

Local cohomology modules are $\mathcal{D}$-modules.

- $K =$ field
- $R = K$-algebra
- $\mathcal{D} = D(R; K)$ is the set of "derivatives" we are allowed to take in $R$ and coefficients are in the field $K$; includes multiplication by elements in $R$

$\mathcal{D}$ stands for the **ring of operators of $R$ over $K$**. The operators include multiplication by elements in $R$
$\implies \mathcal{D}$ is an $R$-algebra, i.e., $\mathcal{D}$-modules are $R$-modules.

## Example

(1) $\mathrm{char}\, K = 0$

$\implies \mathcal{D}_S = D(S;K)$ is the Weyl algebra $K\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\rangle$ or, as an $S$-algebra, $\mathcal{D}_S = S\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\rangle$.

(2) $\mathrm{char}\, K = p > 0$

$\implies \mathcal{D}_S$ is strictly larger than the Weyl algebra – must include the divided powers $\partial_i^p = \frac{1}{p!}\frac{\partial^p}{\partial x_i^p}$

(3) $R = S/J$

$\implies \mathcal{D} = D(R;K) = \frac{\mathcal{I}(J)}{J\mathcal{D}_S}$, where $\mathcal{I}(J)$ denotes the **idealizer of $J$**, the set of operators $\delta \in \mathcal{D}_S$ such that $\delta(J) \subseteq J$

- $\partial_i^t = K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$-linear maps $\frac{1}{t!}\frac{\partial^t}{\partial x_i^t} : R \to R$ where $x_i^u \mapsto \binom{u}{t} x_i^{t-u}$, called **divided powers**
- monomial notation $\mathbf{x}^{\mathbf{a}}\underline{\partial}^{\mathbf{t}} = x_1^{a_1} \cdots x_n^{a_n} \partial_1^{t_1} \cdots \partial_n^{t_n}$

**Theorem (BMW 2015)**

*If $R = S/I_\Delta$ is a Stanley-Reisner ring whose simplicial complex is a $T$-space then $\mathcal{D}$ is generated as an $R$-algebra by operators of the form $x_i \partial_i^t$.*

W. Traves 1999: $\delta = x_i \partial_i^t \implies \delta \in \mathcal{D}$
We show: $T$-space $\implies$ the converse

# Holonomicity

Lyubeznik modified and applied Bavula's definition of holonomicity to characteristic-freely prove the local cohomology modules over a polynomial ring over a field have finitely many associated primes:

$\mathcal{D}_S$ has a filtration of $K$-vector spaces $K = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ where for each $j \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{F}_j = K \cdot \{\mathbf{x}^{\mathbf{a}} \underline{\partial}^{\mathbf{t}} \mid a_1 + \cdots + a_n + t_1 + \cdots t_n \leq j\},$$

called the **Bernstein filtration**.

Recall: If $R = S/J$ then $\mathcal{D} = \frac{\mathcal{I}(J)}{J\mathcal{D}_S}$

$\implies \mathcal{G}_j = \frac{\mathcal{F}_j \cap \mathcal{I}(J)}{J\mathcal{D}_S}$ gives a filtration $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots$ on $\mathcal{D}$.

For $R = S/I_\Delta$, $\mathcal{D} = R\langle x_i \partial_i^t \mid 1 \leq i \leq n, t \geq 0\rangle$

$$\implies \mathcal{G}_j = K \cdot \{\mathbf{x}^{\mathbf{a}} \underline{\partial}^{\mathbf{t}} \mid a_1 + \cdots + a_n + t_1 + \cdots t_n \leq j \text{ and for each } i,\ a_i \geq t_i\};$$

we call $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots$ the **Bernstein filtration on $R$**.

### Definition (Bavula 2009; Lyubeznik 2010; BMW 2015)

A $\mathcal{D}$-module $N$ is **holonomic** means there exists an ascending chain of $K$-modules $N_0 \subset N_1 \subset \cdots$ (called a $K$-**filtration**) satisfying

(i) $\cup_i N_i = N$ and

(ii) for all $i$ and $j$, $\mathcal{G}_j N_i \subset N_{i+j}$,

such that for all $i$, $\dim_K N_i \leq C i^{\dim R}$ for some constant $C$.

### Theorem (BMW 2015)

*Every holonomic $\mathcal{D}$-module has finite length.*

**Theorem (BMW 2015)**

*Suppose $R$ is a Stanley-Reisner ring over a field and its simplicial complex is a $T$-space. Then for all $f \in R$, the localized ring $R_f$ is holonomic.*

**Corollary**

*The local cohomology modules $H^j_I M$ over $R$ have finitely many associated primes.*

# More questions

(1) Does $K$ have to be a field?

(2) Is there an example of a non- $T$-space where the result fails?

Questions from the audience?

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