Toric and tropical Bertini theorems in positive characteristic

Joint Math Meetings
AWM Workshop: Women in Commutative Algebra (WiCA)

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Thank-you to the organizers, and for the invitation to speak!

- Joint w/ F. Gandini, M. Hering, D. Maclagan, F. Mohammadi, J. Rajchgot, J. Yu (GHMMRWY). This project arose from the Women in Commutative Algebra workshop at BIRS 2019.
- Goal: Prove a toric Bertini theorem in arbitrary characteristic.
- Application: Prove a tropical Bertini theorem in arbitrary characteristic and use it to extend a connectedness result by Maclagan and Yu.

Introduction: What is a Bertini theorem?

A **Bertini theorem** states that if a variety X has property $\mathcal{P}$, then so will generic hyperplane sections of X.

Examples:

- Bertini: A general hyperplane section of an irreducible variety of dimension $d \geq 2$ is again irreducible.
- Bertini: In characteristic 0, a general hyperplane section of a smooth variety is again smooth.
- Fuchs+Mantova+Zannier (FMZ) 2018 (Toric Bertini theorem): In characteristic 0, a torus section of an irreducible subvariety of an algebraic torus of dimension $d \geq 2$ is irreducible.

About the toric Bertini theorem:

- The proof of FMZ's toric Bertini theorem uses resolution of singularities, so relies on the characteristic being 0.
- FMZ specifies a finite set of subtori to avoid.
- We prove a toric Bertini theorem in arbitrary characteristic that states that the set of "good" tori is dense, rather than Zariski open.
- FMZ relies on a **PB property** hypothesis that we also need for our result. We do not know if the result is true without it.

Theorem (Toric Bertini theorem; GHMMRWY 2021)

Let $\mathbb{k}$ be an algebraically closed field of arbitrary characteristic. Let $X \subseteq (\mathbb{k}^*)^n$ be an irreducible d-dimensional subvariety where $d \geq 2$ and let $\pi : (\mathbb{k}^*)^n \to (\mathbb{k}^*)^d$ be a morphism such that $\pi|_X$ is dominant and finite. Suppose the pullback of $\pi|_X$ along any isogeny is irreducible. Then for every $1 \leq r \leq d-1$ the set of r-dimensional subtori $T \subseteq (\mathbb{k}^*)^d$ with $\pi^{-1}(\boldsymbol{\theta} T) \cap X$ irreducible for all $\boldsymbol{\theta} \in (\mathbb{k}^*)^d$ is dense in the Grassmannian $\mathrm{Gr}(r, d)$.

Reductions

Important reductions:

- Any pathological torus T will contain a one-dimensional pathological torus, so we may consider only the case where T is one-dimensional.
- Any variety is birational to a hypersurface so we reduce to the case were X is a hypersurface in $(\mathbb{k}^*)^d \times \mathbb{A}^1$.
- We reduce to the case where π is projection onto the first d coordinates.

With these reductions we have $X = \mathcal{V}(f)$ where $f \in \Bbbk[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, y]$.

If $T \subseteq (\Bbbk^*)^d$ is a one-dimensional torus parametrized by $z \mapsto (z^{n_1}, \ldots, z^{n_d})$ and $\boldsymbol{\theta} \in (\Bbbk^*)^d$ then $\pi^{-1}(\boldsymbol{\theta} T) \cap X$ is isomorphic to the variety given by the specialization

$$f(t_1 \cdots, t_d, y) \mapsto f(\theta_1 x^{n_1}, \ldots, \theta_d x^{n_d}, y) \in \Bbbk[x^{\pm 1}, y].$$

The problem reduces to showing that if $f(t_1, \ldots, t_d, y)$ is irreducible, then so is $f(\theta_1 x^{n_1}, \ldots, \theta_d x^{n_d}, y)$ for all choices of $\boldsymbol{\theta} \in (\Bbbk^*)^d$ and *most* choices of $\mathbf{n} \in \mathbb{Z}^d$.

To prove it, we first factor f over a generalized Puiseux series field, **and this is the heart of the proof.**

We show that the factors are preserved under the specialization.

Then we show that for most $\mathbf{n}$, none of the factors of the specialization are polynomials.

Puiseux series

Puiseux: For $\mathbb{k}$ an algebraically closed field of characteristic 0, the **field of Puiseux series** given by

$$\mathbb{k}\{\{x\}\} = \bigcup_{n \geq 1} \mathbb{k}(\!(x^{\frac{1}{n}})\!)$$

is algebraically closed.

We wish to find a multivariate field of Puiseux series that is algebraically closed in any characteristic.

Example: In characteristic p, Chevalley observed the Artin-Shreier polynomial $x^p - x - t^{-1}$ has no Puiseux series root. Abyankar showed $\sum_{j \geq 0} t^{\frac{1}{p^j}}$ is a root.

In characteristic 0, Puiseux series all have a common denominator in the exponents. In characteristic p we must relax this condition to allow for arbitrarily large powers of p in the denominators.

In 1968 Rayner introduced the notion of a **field family** to show that in characteristic p, the collection

$$\left\{ \sum c_a t^a \mid \text{there is } N > 0 \text{ such that } a \in \bigcup_{j \geq 0} \frac{1}{Np^j}\mathbb{Z} \text{ for all } a \text{ such that } c_a \neq 0 \text{ and } \{a \mid c_a \neq 0\} \text{ is well-ordered} \right\}$$

forms an algebraically closed field.

Field families give conditions on the exponents (supports) of series in order for the collection of series to form a field.

Saavedra (2017) used field families to find a field of multivariate Puiseux series in characteristic p.

The requirement is that the supports lie in

$$\bigcup_{j\geq 0} \frac{1}{Np^j}\mathbb{Z}^d \cap C$$

for some $N > 0$ and some pointed cone C in the direction of a vector $\mathbf{w}$, whose coordinates are linearly independent over $\mathbb{Q}$, and that the supports are well-ordered with respect to the dot product with $\mathbf{w}$.

What is the limitation of Saavedra's result for us?

The specialization map

$$\sum c_{\mathbf{u}} \mathbf{t}^{\mathbf{u}} \mapsto \sum c_{\mathbf{u}} \boldsymbol{\theta}^{\mathbf{u}} x^{\mathbf{n}\cdot\mathbf{u}}$$

may not be well-defined.

Example: Let char $\mathbb{k} = 2$,

$$\alpha = \sum_{j \geq 1} t_1^{1-\frac{1}{2^j}} t_2^{\frac{1}{2^j}},$$

$\mathbf{n} = (1,1)$, and $\boldsymbol{\theta} = (1,1)$. The vector $\mathbf{w}$ is any vector close to $\mathbf{n}$ and the cone is on one side of the hyperplane given by $\mathbf{w}$.

The specialization map takes $\alpha \mapsto \sum_{j \geq 1} x^{1-\frac{1}{2^j}} x^{\frac{1}{2^j}} = \sum_{j \geq 1} x$, which is not defined.

We introduce the notion of p-discreteness with respect to $\mathbf{w}$, whose coordinates are linearly independent over $\mathbb{Q}$.

Let $\mathbb{R}^d_{li}$ denote the set of vectors whose coordinates are linearly independent with respect to $\mathbb{Q}$.

Definition (GHMMRWY 2021)

Fix $\mathbf{w} \in \mathbb{R}^d_{li}$. A set $A \subseteq \mathbb{Q}^d$ is ***p*-discrete with respect to w** means the following axioms hold:

Definition (p-discrete, cont.)

(a) There exists an open pointed rational polyhedral cone σ, $\mathbf{w} \in \sigma$, such that $\{\mathbf{w}' \cdot \mathbf{a} \mid \mathbf{a} \in A\}$ is well-ordered for all $\mathbf{w}' \in \sigma \cap \mathbb{R}^d_{li}$.

This is stronger than Saavedra's condition of well-ordering. We need it to show the specializations are well-defined.

Definition (p-discrete, cont.)

(b) There is $N > 0$, $\boldsymbol{\gamma} \in \mathbb{Q}^d$, and a pointed rational polyhedral cone C, $\mathbf{w} \in \operatorname{int}(C^\vee)$, such that $A \subseteq (\boldsymbol{\gamma} + C) \cap \left(\bigcup_{j \geq 0} \frac{1}{Np^j}\mathbb{Z}^d\right)$.

We need the translation condition to ensure that our field of Puiseux series is a field.

Definition (p-discrete, cont.)

(c) Any sequence $\{\mathbf{a}_i\} \subset A$ converges in $\mathbb{Q}^d$ if $\{\mathbf{w} \cdot \mathbf{a}_i\}$ converges in $\mathbb{R}$.

This is a technical condition needed to show p-discrete supports form a field family.

Definition (p-discrete, cont.)

(d) For all $\gamma' \in \mathbb{Q}^d$ there is an open pointed rational polyhedral cone $\sigma_{\gamma'}$, $\mathbf{w} \in \sigma_{\gamma'}$, such that for all $\mathbf{w}' \in \sigma_{\gamma'} \cap \mathbb{R}^d_{li}$, we have

$$\{\mathbf{a} \in A \mid \mathbf{w}' \cdot \mathbf{a} > \mathbf{w}' \cdot \gamma'\} = \{\mathbf{a} \in A \mid \mathbf{w} \cdot \mathbf{a} > \mathbf{w} \cdot \gamma'\} \text{ and}$$
$$\{\mathbf{a} \in A \mid \mathbf{w}' \cdot \mathbf{a} < \mathbf{w}' \cdot \gamma'\} = \{\mathbf{a} \in A \mid \mathbf{w} \cdot \mathbf{a} < \mathbf{w} \cdot \gamma'\}.$$

This condition provides narrow cones where the monomials with those supports do not get cancelled under the specialization. It is needed to show that no factor of f is a polynomial.

Applications and other questions

What is a tropical variety? A tropical variety turns an algebraic variety into a polyhedral complex.

Part of the Structure Theorem in tropical geometry states that given an irreducible variety X in $(\mathbb{k}^*)^n$ of dimension d, the tropicalization $\text{trop}(X)$ of X is d-connected through codimension one, i.e., it remains closed after removing $d-1$ closed facets.

Connectivity through codimension one is essential for computations in tropical geometry.

Maclagan+Yu (2019) proved a stronger statement in characteristic 0, using the tropcial Bertini theorem.

The tropical Bertini theorem in characteristic 0 relies on FMZ's toric Bertini theorem.

Using our generalized toric Bertini theorem, it holds in arbitrary characteristic:

Theorem (Tropical Bertini theorem; GHMMRWY 2021)

Let $X \subseteq (\Bbbk^*)^n$ be an irreducible d-dimensional variety, for $d \geq 2$ with $\Bbbk$ algebraically closed and $\mathbb{Q}$ in its value group. Then the set of rational affine hyperplanes $H \subseteq \mathbb{R}^n$ for which $\mathrm{trop}(X) \cap H$ is the tropicalization of an irreducible variety is dense in the Euclidean topology on $\mathbb{P}^n_{\mathbb{Q}}$.

Theorem (GHMMRWY 2021)

Let $\mathbb{k}$ be a field that is either algebraically closed, complete, or real closed with convex valuation ring. Let $X \subseteq (\mathbb{k}^*)^n$ be an irreducible d-dimensional subvariety, let Σ be a pure d-dimensional rational polyhedral complex with support $|\Sigma| = \mathrm{trop}(X)$, and let ℓ denote the dimension of the lineality space of Σ. Then Σ is $(d - \ell)$-connected through codimension 1.

Other questions:

(1) Can we remove the PB property hypothesis in the toric Bertini theorem?

(2) Our field of generalized Puiseux series contains the algebraic closure of $\mathbb{k}(t_1, \ldots, t_d)$ and so does Saavedra's field. What is the algebraic closure of $\mathbb{k}(t_1, \ldots, t_d)$?

Thank-you!