Toric varieties given by principal 2-minor ideals

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Thank-you for the invitation to speak!

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Introduction

Let K be an algebraically closed field and $X = (x_{ij})$ an $n \times n$ matrix of variables.

A **principal** t**-minor** of X is a $t \times t$ minor whose row and column indices are the same, i.e., it is symmetric about the main diagonal of X.

We have a **principal** t**-minor ideal** $\mathfrak{P}_t$ in the polynomial ring $K[X]$, which is generated by the principal t-minors of X.

Historically, ideals generated by collections of minors of a generic matrix have been well-studied, including determinantal ideals and Pfaffian ideals.

However, not much is known about principal t-minor ideals. We are motivated to study them because of their applications in algebraic statistics.

In this project, we are particularly interested in $\mathfrak{P}_2$.

Theorem (Ene+Qureshi 2013, W 2015)

The ideal $\mathfrak{P}_2$ is toric, that is, it is prime and generated by binomials.

Ideals generated by binomials have applications in integer programming. More interestingly, the vanishing set of a toric ideal is a toric variety.

A **toric variety** is an irreducible variety V that contains an algebraic torus as a Zariski open subset, such that the group action of the torus on itself extends to an action on V.

Toric varieties provide a wide class of non-trivial examples where various theorems in algebraic geometry can be tested, and much about their structure can be deduced using techniques in convex geometry.

Monomial maps

Toric varieties can be defined using a monomial map of an algebraic torus.

Let $\mathcal{A} = \{\mathbf{m}_1, \ldots, \mathbf{m}_s\} \subseteq \mathbb{Z}^d$ and define the map

$$\begin{aligned} \Phi_{\mathcal{A}} : (K^*)^d &\to (K^*)^s \subseteq K^s \\ \mathbf{t} &\mapsto (\mathbf{t}^{\mathbf{m}_1}, \ldots, \mathbf{t}^{\mathbf{m}_s}) \end{aligned}$$

where for $\mathbf{t} = (t_1, \ldots, t_d) \in (K^*)^d$ and $\mathbf{m} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$, $\mathbf{t}^{\mathbf{m}} = t_1^{a_1} \cdots t_d^{a_d}$.

The **affine toric variety** $Y_{\mathcal{A}}$ is the Zariski closure in K^s of the image of $\Phi_{\mathcal{A}}$, which is its torus.

The **projective toric variety** $X_{\mathcal{A}}$ is the Zariski closure of the image of the composition map (which is its torus)

$$(K^*)^d \xrightarrow{\Phi_{\mathcal{A}}} K^s \xrightarrow{\eta} \mathbb{P}^{s-1},$$

where η is the quotient map $\eta : K^s \to (K^s \setminus \{\mathbf{0}\})/K^* = \mathbb{P}^{s-1}$.

Since $\mathfrak{P}_2$ is homogeneous, the algebraic set $V(\mathfrak{P}_2)$ can be viewed as an affine or projective toric variety. In this talk we will mainly regard $V(\mathfrak{P}_2)$ as an affine toric variety.

We first find the finite set $\mathcal{A}$ of integer vectors that gives the correct monomial map to define $V(\mathfrak{P}_2)$.

To do this we need to know how to find the defining ideal for an affine toric variety $Y_{\mathcal{A}}$.

Let L denote the kernel of the map

$$\hat{\Phi}_{\mathcal{A}} : \mathbb{Z}^s \to \mathbb{Z}^d$$
$$\mathbf{e}_i \mapsto \mathbf{m}_i$$

and let $B = \{\mathbf{b}^1, \ldots, \mathbf{b}^r\}$ be a basis for L.

Then the defining ideal for $Y_{\mathcal{A}}$ is

$$\mathbf{I}(Y_{\mathcal{A}}) = \left\langle \prod_{b_i^j > 0} x_i^{b_i^j} - \prod_{b_i^j < 0} x_i^{-b_i^j} \,\middle|\, \mathbf{b}^j = (b_1^j, \ldots, b_s^j) \in B \right\rangle : \langle x_1 \cdots x_s \rangle^\infty.$$

If $\mathbf{I}(Y_{\mathcal{A}})$ is homogeneous then $\mathbf{I}(Y_{\mathcal{A}}) = \mathbf{I}(X_{\mathcal{A}})$.

Notation

For vectors in $\binom{n+1}{2}$ dimensions, we use the indexing $11, 12, \ldots,$ $1n, 22, \ldots 2n, \ldots, nn$ for the components and for vectors in n^2 dimensions we use the indexing $11, 12, \ldots, 1n, 21, 22, \ldots, 2n, 31,$ $\ldots, nn$.

Proposition (BRSW)

$\mathbf{I}(Y_{\mathcal{A}_n}) = \mathbf{I}(X_{\mathcal{A}_n}) = \mathfrak{P}_2$, *where*

$$\mathcal{A}_n = \{\mathbf{e}_{ij} \mid 1 \leq i \leq j \leq n\} \\ \cup \{\mathbf{e}_{ii} - \mathbf{e}_{ij} + \mathbf{e}_{jj} \mid 1 \leq i \leq j \leq n\} \subseteq \mathbb{Z}^{\binom{n+1}{2}}.$$

Example

For $n = 2$ we have $\mathcal{A}_2 = \{\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{22}, \mathbf{e}_{11} - \mathbf{e}_{12} + \mathbf{e}_{22}\} \subseteq \mathbb{Z}^3$ and the monomial map

$$\Phi_{\mathcal{A}_2} : (K^*)^3 \to (K^*)^4 \subseteq K^4$$
$$(t_{11}, t_{12}, t_{22}) \mapsto (t_{11}, t_{12}, t_{22}, t_{11} t_{12}^{-1} t_{22}).$$

The kernel of $\hat{\Phi}_{\mathcal{A}_2}$ is $L = \mathbb{Z}\{(1, -1, -1, 1)\}$ and

$$\mathbf{I}(Y_{\mathcal{A}_2}) = \langle x_{11}x_{22} - x_{12}x_{21} \rangle : \langle x_{11}x_{12}x_{21}x_{22} \rangle^\infty = \mathfrak{P}_2 = \mathbf{I}(X_{\mathcal{A}_2}).$$

Cones

So far we have two ways to describe an affine toric variety in K^s:

1. $V(I) \subseteq K^s$ for some toric ideal $I \subseteq K[x_1, \ldots, x_s]$ and
2. $Y_{\mathcal{A}}$ for some finite set $\mathcal{A} \subseteq \mathbb{Z}^d$.

We now illustrate a third way to describe an affine toric variety, using cones.

The torus $T = \mathrm{im}(\Phi_{\mathcal{A}}) \subseteq Y_{\mathcal{A}}$ comes equipped with dual lattices $M = \mathbb{Z}\mathcal{A}$ and $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, both with rank equal to $\dim T$.

M is the **character lattice** of T, consisting of the group of group homomorphisms $\chi^{\mathbf{m}} : T \to K^*$ for $\mathbf{m} \in M$.

N is the group of **one-parameter subgroups** of T, the group of group homomorphisms $\lambda^{\mathbf{u}} : K^* \to T$ for $\mathbf{u} \in N$. We have $N \otimes_{\mathbb{Z}} K^* \cong T$ via the map $\mathbf{u} \otimes t \mapsto \lambda^{\mathbf{u}}(t)$ and thus it is customary to write $T = T_N$.

An affine toric variety can also be described as $\operatorname{Spec} K[S_\sigma]$ for some rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

$S_\sigma = \sigma^\vee \cap M$, where $\sigma^\vee$ is the **dual cone** of σ,

$$\sigma^\vee = \{\mathbf{m} \in M \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \mathbf{m}, \mathbf{u} \rangle \geq 0 \text{ for all } \mathbf{u} \in \sigma\}.$$

$K[S_\sigma] \subseteq K[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ is the K-subalgebra generated by monomials whose exponent vectors are in S_σ.

Theorem (BRSW)

Let

$$\sigma_n^\vee = \mathrm{cone}(\mathcal{A}_n) = \left\{ \sum_{\mathbf{m}\in\mathcal{A}_n} \lambda_{\mathbf{m}}\mathbf{m} \;\middle|\; \lambda_{\mathbf{m}} \geq 0 \text{ for all } \mathbf{m} \in \mathcal{A}_n \right\}.$$

Then $\sigma_n = (\sigma_n^\vee)^\vee$ *is strongly convex and rational. Therefore,* $\operatorname{Spec} K[S_{\sigma_n}]$ *is normal.*

To show $Y_{\mathcal{A}_n} \cong \operatorname{Spec} K[S_{\sigma_n}]$ we need one more ingredient.

A **Hilbert basis** $\mathcal{H}$ for S_σ is a unique finite subset that minimally generates the semigroup S_σ.

A Hilbert basis for S_σ exists when σ is full-dimensional.

In that case, $Y_{\mathcal{H}} \cong \operatorname{Spec} K[S_\sigma]$ and $Y_{\mathcal{H}} \subseteq K^{|\mathcal{H}|}$ is the most efficient affine embedding of $\operatorname{Spec} K[S_\sigma]$.

Theorem (BRSW)

If $\sigma_n^\vee = \operatorname{cone}(\mathcal{A}_n)$ *then* $\sigma_n = (\sigma_n^\vee)^\vee = \operatorname{cone}(\mathcal{B}_n)$, *where* $\mathcal{B}_n = \cup_{i=1}^n \mathcal{B}_n^i$ *and*

$$\mathcal{B}_n^i = \left\{ \mathbf{e}_{ii} + \sum_{\mathbf{e}\in E} \mathbf{e} \;\middle|\; E \subseteq \{\mathbf{e}_{1i}, \ldots, \mathbf{e}_{i-1,i}, \mathbf{e}_{i,i+1} \ldots, \mathbf{e}_{in}\} \right\} \subseteq \mathbb{Z}^{\binom{n+1}{2}}$$

Example

$\mathcal{B}_2 = \{\mathbf{e}_{11}, \mathbf{e}_{11} + \mathbf{e}_{12}, \mathbf{e}_{22}, \mathbf{e}_{12} + \mathbf{e}_{22}\}$.

It can be shown that $\dim \sigma_n = \binom{n+1}{2}$ and so S_{σ_n} has a Hilbert basis.

Theorem (BRSW)

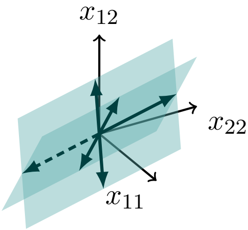
$\mathcal{A}_n$ is the Hilbert basis for S_{σ_n}. Therefore, $Y_{\mathcal{A}_n} \cong \mathrm{Spec}\, K[S_{\sigma_n}]$.

While we have not yet proved that $\mathcal{B}_n$ is the minimal generating set for σ_n (which can then be used to determine whether $Y_{\mathcal{A}_n}$ is smooth), the vectors in $\mathcal{B}_n$ appear again in the facet presentation of the polytope associated to the projective toric variety $X_{\mathcal{A}_n}$.

Other results and questions

We also explored the projective toric variety $X_{\mathcal{A}_n}$. We

1. showed the dimension of the lattice polytope P_n associated to $X_{\mathcal{A}_n}$ is equal to $\binom{n+1}{2} - 1$.
2. computed the normal fan to the lattice polytope P_n. As a consequence we found $X_{\mathcal{A}_n}$ is normal, separated, and compact.



Q: Is $Y_{\mathcal{A}_n}$ smooth?

Evidence (Macaulay2) shows that $\mathcal{B}_n$ is the minimal generating set for the cone σ_n. If this is true, then it implies $Y_{\mathcal{A}_n}$ is not smooth. Once we verify this, we would like to find the singularities of $Y_{\mathcal{A}_n}$.

Thank-you!