Defining equations for matroid varieties – using the Grassmann-Cayley algebra

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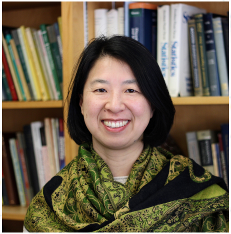
- Joint w/ Jessica Sidman (MHC) and Will Traves (USNA).



- Goal: To find defining equations for matroid varieties.

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Introduction

What is a matroid?

Q: Which collections of 3 columns form a basis for the column space of A?

$$A = \begin{pmatrix} -1 & 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 0 & -2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

How can we tell?

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Could row reduce A first.

$$\begin{pmatrix} -1 & 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 0 & -2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \to \text{rref } A = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Preferred way: Look for non-zero **3-minors** of A (or of rref A).

Fact: Columns of an $r \times r$ matrix are linearly independent if and only if the determinant of the matrix is non-zero.

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Number the columns of A 1 through 5. Here are the 3-minors, according to their column indices:

$$\begin{array}{ccc} \{1,2,3\} \to 0 & \{1,2,4\} \to -6 & \{1,2,5\} \to -12 \\ \{1,3,4\} \to -9 & \{1,3,5\} \to -18 & \{1,4,5\} \to -9 \\ \{2,3,4\} \to 3 & \{2,3,5\} \to -6 & \{2,4,5\} \to -11 \\ \{3,4,5\} \to 0 & & \end{array}$$

The **matroid** $\mathcal{M}_A$ on A is given by the set:

$\mathcal{B} = \{\text{all 3-tuples except } \{1,2,3\} \text{ and } \{3,4,5\}\} \subset \{1,\ldots,5\}$

The sets in $\mathcal{B}$ are called **bases** for the matroid $\mathcal{M}_A$.

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Matroids are a generalization of the collections of linearly independent columns of a matrix.

Applications: Where are matroids used?

- *Combinatorial optimization:* artificial intelligence, machine learning, software engineering; example: travelling salesman problem
- *Coding theory:* error correcting, data compression, cryptography
- *Network theory:* particle physics, biology, social networks; example: bridges of Königsberg problem

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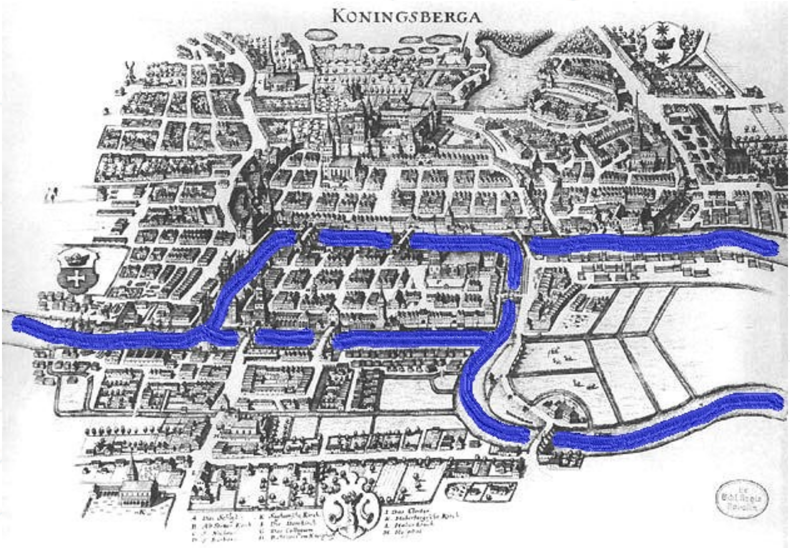

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Problem

Defining equations for matroid varieties

Q: Given a matroid on a matrix A, what other matrices have the same matroid as A? (e.g., rref A)

Consider this **generic matrix**:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \end{pmatrix}$$

We want all possible x-values that will give the same matroid (i.e. same columns as bases) as A.

Defining equations are a system of equations in the x's whose solution set minimally cuts out all of the matrices with matroid $\mathcal{M}_A$.

Call this solution set $\mathcal{V}_A$, the **matroid variety** on A.

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Mnëv (1985), Sturmfels (1989), Knutson, Lam, & Speyer (2013): **All "hell" breaks loose when we try to find defining equations for $\mathcal{V}_A$.**

Why? Two reasons:

1. The *Zariski topology* (beyond the scope of this talk).
2. The equations are not obvious (as we shall see).

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Paradigm shift

Matroids as point configurations

Idea: Think of the columns of A as points in the plane.

The columns of the matrix A are vectors in 3-space, that span lines through the origin.

If we cut these lines with a plane then the lines become points in the plane.

Why would we do this?

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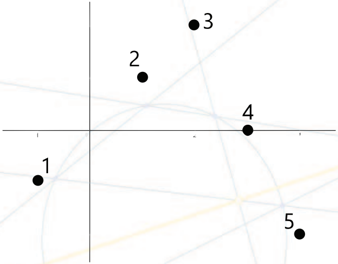

Q: What do you notice about the points in relation to the matroid $\mathcal{M}_A$?

Points 1, 2, and 3 are collinear if and only if $\{1, 2, 3\}$ is not a basis for $\mathcal{M}_A$. We say the **bracket** $[123]$ vanishes, or equals 0.

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The bracket is shorthand for determinant:

$$[123] = 0 \iff \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = 0$$

This is a defining equation in the x's!

Q: What is another defining equation for $\mathcal{V}_A$?

Are there others? ???

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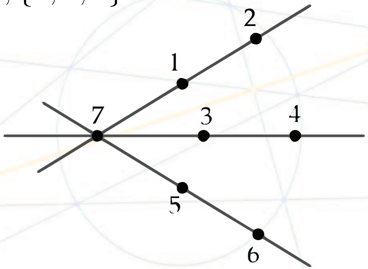
Example (Ford 2013):

$\mathcal{M}_{\text{pencil}}$ = matroid on a 3×7 matrix with *non*bases $\{1,2,7\}, \{3,4,7\}, \{5,6,7\}$



Q: Which brackets vanish?

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All hell breaks loose: $[134][256] - [234][156] = 0$ is also a defining equation for $\mathcal{V}_{\text{pencil}}$. How could we have known this?!

Ans: The **Grassmann-Cayley algebra**.

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Tool

Grassmann-Cayley algebra

The Grassmann-Cayley algebra was invented to do *synthetic projective geometry*.

We do arithmetic on the points (vectors) themselves, and obtain expressions in the brackets.

There are two operations:

join ($\vee$): refers to the line passing through points

The line joining 1 and 2 is $1 \vee 2$, or 12.

Three points joined makes a bracket, and three collinear points make the bracket vanish.

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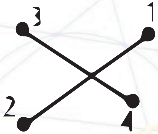
meet ($\wedge$): refers to the intersection of two lines



The meet of the lines $1 \vee 2$ and $3 \vee 4$ is $(1 \vee 2) \wedge (3 \vee 4)$, or $12 \wedge 34$.

The meet operation uses *shuffle products*, which also produce expressions in the brackets.

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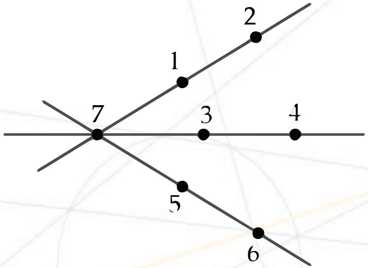

We have $((1 \vee 2) \wedge (3 \vee 4)) \vee 5 \vee 6 = 0$, because these three points are collinear. Here's how to apply the shuffles:

$$\begin{aligned}(12 \wedge 34) \vee 56 &= ([134]2 - [234]1)56 \\ &= [134][256] - [234][156] = 0\end{aligned}$$

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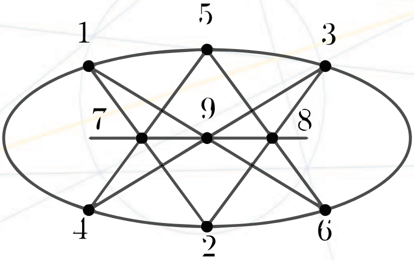
Example: **Pascal's theorem** says if six points on a conic are joined in the way illustrated below, then the resulting intersection points (7, 8, and 9) are collinear.



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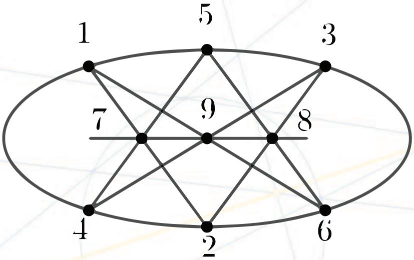

We have a matroid $\mathcal{M}_{\text{Pascal}}$ corresponding to this configuration of points. Let's find defining equations for $\mathcal{V}_{\text{Pascal}}$.

Q: Which brackets vanish?

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Theorem (Sidman, Traves, W)

The defining equations for the matroid variety $\mathcal{V}_{Pascal}$ include at least one quartic, three independent cubics, and three independent quadrics in the brackets, besides the vanishing brackets $[127], [238], [349], [457], [568], [169], [789]$.

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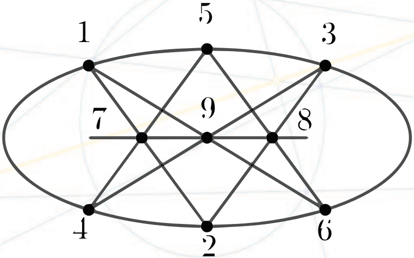
Finding the quartic: The statement of Pascal's theorem says we have $(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 61) = 0$.



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Apply the shuffle products:

$$(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 61)$$
$$= ([145]2 - [245]1) \vee ([256]3 - [356]2) \vee ([361]4 - [461]3)$$

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Now “foil”:

$$
\begin{aligned}
&(([145]2 - [245]1) \vee ([256]3 - [356]2)) \vee ([361]4 - [461]3) \\
= &([145][256]23 - [145][356]22 - [245][256]13 + [245][356]12) \\
&\quad \vee ([361]4 - [461]3)
\end{aligned}
$$

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Finish "foiling":

$$
\begin{aligned}
&([145][256]23 - [145][356]22 - [245][256]13 + [245][356]12) \\
&\quad \vee ([361]4 - [461]3) \\
&= [145][256][361][234] - [145][256][461][233] \\
&\quad - [145][356][361][224] + [145][356][461][223] \\
&\quad - [245][256][361][134] + [245][256][461][133] \\
&\quad + [245][356][361][124] - [245][356][461][123]
\end{aligned}
$$

Q: What happens when two numbers share a bracket?

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$$[233] \leftrightarrow \begin{vmatrix} x_{12} & x_{13} & x_{13} \\ x_{22} & x_{23} & x_{23} \\ x_{32} & x_{33} & x_{33} \end{vmatrix}$$

Q: What happens to a matrix with two repeated columns?

It means we get a bunch of cancelling...

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$$
\begin{aligned}
&[145][256][361][234] - [145][256][461][233] \\
&\quad - [145][356][361][224] + [145][356][461][223] \\
&\quad - [245][256][361][134] + [245][256][461][133] \\
&\quad + [245][356][361][124] - [245][356][461][123] \\
&= [145][256][361][234] - [245][256][361][134] \\
&\quad + [245][356][361][124] - [245][356][461][123] = 0
\end{aligned}
$$

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Conclusion:

- Finding defining equations for matroid varieties is HARD.
- The Grassmann-Cayley algebra lets us find *some* of the defining equations, by using the geometry of the matroids when thought of as point configurations.

Thank-you!

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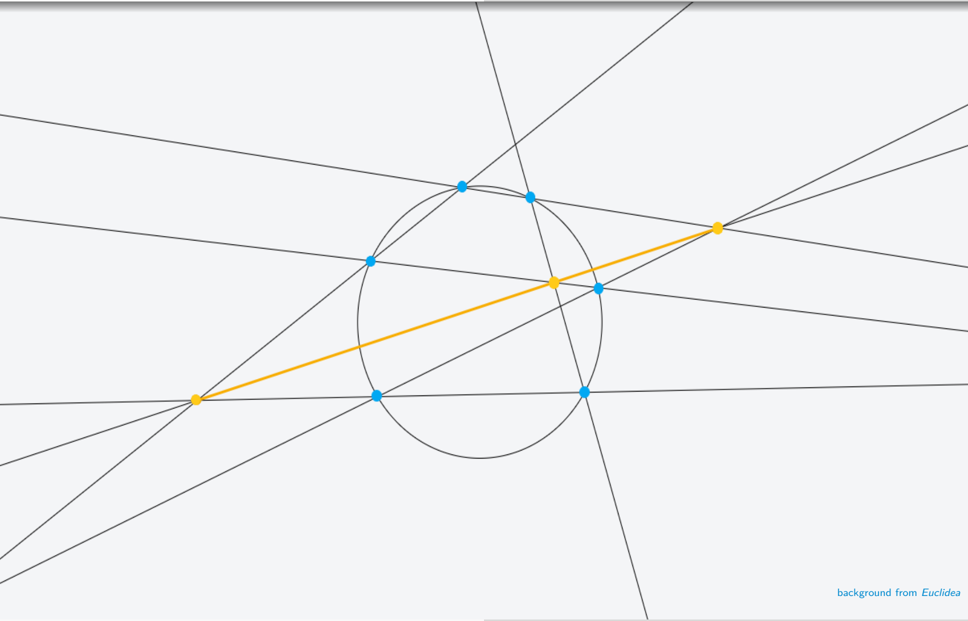

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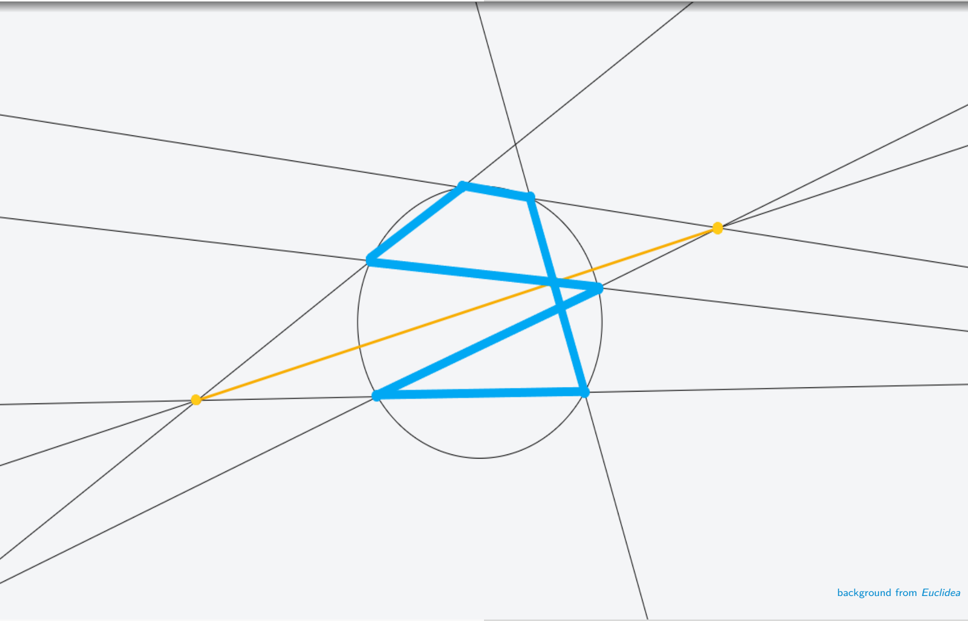

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