Calculus II (M102)

Spring 2021 Mod 1

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1 Intro to integration techniques

§6.1 Antiderivatives graphically and numerically

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Indefinite integrals

Recall: **definite integrals**

Example

Evaluate $\int_0^2 x^2 \; dx$.

Soln:

$$\int_0^2 x^2 \; dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{(2)^3}{3} - \frac{(0)^3}{3} = \frac{8}{3}.$$

Notes:

- No "$+C$".
- Answer is a number, not a function.
- Vertical bar indicates the bounds on the integral have yet to be plugged in.
- Style: Plugged in values are in parentheses (this makes it easier to read and catch mistakes). Answer is written as a fraction, and not a decimal (eliminates rounding errors).

Recall: **The Fundamental Theorem of Calculus**

Theorem

If f is continuous on the interval $[a, b]$, and $f(t) = F'(t)$, then

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

$\star$ For a refresher, see §5.3.

Recall: **indefinite integrals**

Example

$\int x^2 \, dx = \frac{x^3}{3} + C$

Notes:

- "$+C$".
- No bounds on the integral symbol.
- Answer is a (family of) function(s), and not a number.

The "$+C$" represents the fact that infinitely many functions can have the same derivative function.

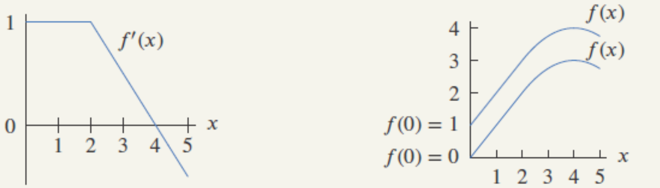


Figure 6.1: Graph of f'

Figure 6.2: Two different f's which have the same derivative f'

Graphic: Example 1 of §6.1.

Notes:

- The derivative function is constant from $0 \leq x \leq 2$. This means the *slope* of $f(x)$ should also be constant.
- The function $f'(x)$ changes from positive to negative at $x = 4$. This means the functions $f(x)$ should have a local max at $x = 4$ (recall: Calc I).
- The difference between the two $f(x)$s is where they begin, or their values for $f(0)$.

Sketching antiderivatives

Example (§6.1 #22)

Using Figure 6.15, sketch a graph of an antiderivative $G(t)$ of $g(t)$ satisfying $G(0) = 5$. Label each critical point of $G(t)$ with its coordinates.

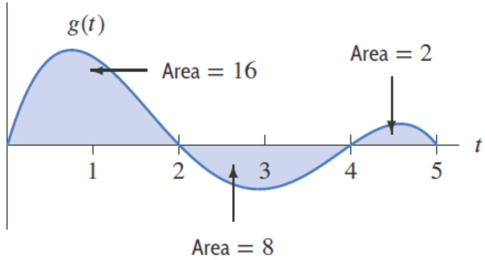
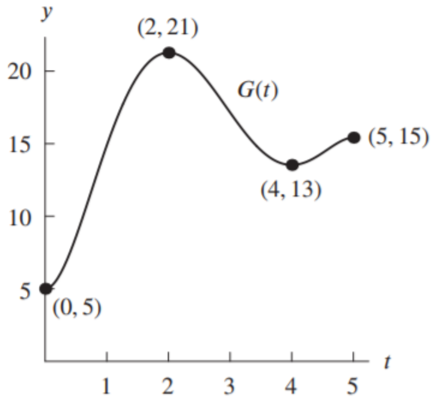


Figure 6.15

Graphic: §6.1 #22

Soln:



Graphic: solution to §6.1 #22

Q: Where are the inflection points?

Families of functions: initial value problems

In practice (a.k.a. real life), we're given information about the derivative of something and we have to find the original function.

Example (Finding the right "+C")

Find $f(x)$, given that:

$$f'(x) = 3x^2 \quad \text{and} \quad \underbrace{f(1) = 3}_{\text{called an } \textbf{initial value}}$$

Soln: Use an indefinite integral first:

$$\int f'(x)\, dx = \int 3x^2\, dx = \underbrace{x^3 + C}_{f(x)}$$

Use the initial value to solve for the correct C:

$$\begin{aligned} f(1) = 1^3 + C = 3 \\ \implies C = 3 - 1 = 2 \end{aligned}$$

The conclusion is that $f(x) = x^3 + 2$.

$\star$ In Differential Equations (M333) it gets more complicated!

1 Intro to integration techniques

Reverse derivative rules

Reverse power rule:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

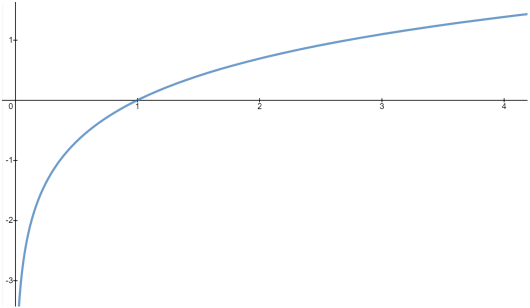
Q: Why can't $n = -1$?

$n = -1$ case:

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

Q: Why do we need absolute values?

Ans: Because x cannot be negative in the natural log function.



Graphic: Natural log function.

Exponential function:

$$\int e^x \, dx = e^x + C$$

Sine & cosine:

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

More:

$$\int \sec^2 x \; dx = \tan x + C$$

$$\int \csc^2 x \; dx = -\cot x + C \quad \text{... etc.}$$

Q: What is $\int 2^x \; dx$?

Later:

$$\int \tan x \; dx \quad (\text{§7.1})$$

$$\int \ln x \; dx \quad (\text{§7.2})$$

Linearity of integrals

Linearity of integrals means the following two conditions hold:

- $\int (f(x) + g(x))\ dx = \int f(x)\ dx + \int g(x)\ dx$
- $\int cf(x)\ dx = c \int f(x)\ dx$, where c is a constant

Q: Where have you seen this property before?

Linearity also holds for definite integrals.

Linearity lets us integrate polynomials.

Example (§6.2 #24)

Find $\int (x^2 - 4x + 7)\, dx$.

Soln:

$$\begin{aligned}\int (x^2 - 4x + 7)\ dx &= \int x^2\ dx + \int (-4x)\ dx + \int 7\ dx \\ &= \int x^2\ dx - 4\int x\ dx + 7\int x^0\ dx \\ &= \left(\frac{x^3}{3} + C_1\right) - 4\left(\frac{x^2}{2} + C_2\right) + 7\left(\frac{x^1}{1} + C_3\right) \\ &= \frac{x^3}{3} - 2x^2 + 7x + C,\end{aligned}$$

where $C_1 + -4C_2 + 7C_3$ gets absorbed into one arbitrary constant, C.

The reverse power rule lets us integrate radicals.

Example (§6.2 #54)

Find the general antiderivative of $\sqrt{x^3} - \frac{2}{x}$.

Soln:

$$\begin{aligned}\int \left(\sqrt{x^3} - \frac{2}{x}\right)\, dx &= \int \sqrt{x^3}\, dx + \int -\frac{2}{x}\, dx \\ &= \int x^{\frac{3}{2}}\, dx - 2\int \frac{1}{x}\, dx \\ &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 2\ln|x| + C \\ &= \frac{2}{5}x^{\frac{5}{2}} - 2\ln|x| + C\end{aligned}$$

Again, the constants from each individual integral get absorbed into one big constant C.

1 Intro to integration techniques

§7.1 Integration by substitution

- Reverse chain rule
- Substitution for definite integrals
- Trick: Linear substitutions

Reverse chain rule

Here's the idea. Recall, the chain rule is used to differentiate compositions of functions:

$$\frac{d}{dx}f(g(x)) = f'\underbrace{(g(x))}_{w} \cdot \underbrace{g'(x)}_{w'}$$

$$= \frac{df}{dw} \cdot \frac{dw}{dx}$$

To integrate,

$$\begin{aligned}\int \left(\frac{d}{dx} f(g(x))\right) \, dx &= f(g(x)) + C \quad \leftarrow \text{ want} \\ &= \int \frac{df}{dw} \cdot \frac{dw}{\cancel{dx}} \, \cancel{dx} \\ &= \int f'(w) \, dw.\end{aligned}$$

Hopefully, it makes the integral easier. The trick is finding out what w is.

Example

$\int 3x^2 \cos\left(x^3\right)\ dx$

Soln: Let $w = x^3 \implies \frac{dw}{dx} = 3x^2$. Match to the original integral:

$$\int f'(w) \cdot \underbrace{3x^2\ dx}_{dw}$$

Q: What is $f'(w)$?

Ans: $\cos w$, since $w = x^3$.

Replace the integral and evaluate:

$$\int \cos w \; dw = \sin w + C$$

Put the xs back:

$$\sin w + C = \sin\left(x^3\right) + C$$

Check: $\frac{d}{dx} \sin\left(x^3\right) = 3x^2 \cos\left(x^3\right)$

Substitution for definite integrals

Method 1: Keep the bounds in terms of x until x is substituted back in.

Example (§7.1 #60)

Evaluate $\int_0^{\frac{1}{2}} \cos(\pi x)\, dx$.

Soln: Let $w = \pi x \implies \frac{dw}{dx} = \pi$. We have

$$\int_0^{\frac{1}{2}} \cos(\pi x)\, dx = \int_{x=0}^{x=\frac{1}{2}} \cos w \cdot \underbrace{\frac{1}{\pi}\, dw}_{\text{solving for } dx}$$

$$= \frac{1}{\pi} \sin w \Big|_{x=0}^{x=\frac{1}{2}}$$

$$= \frac{1}{\pi} \sin(\pi x) \Big|_0^{\frac{1}{2}}$$

$$= \frac{1}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right)$$

$$= \frac{1}{\pi}.$$

Method 2: Change the bounds on the integral to match the new variable w, then integrate normally.

Example (§7.1 #64)

Evaluate $\int_{-1}^{e-2} \frac{1}{t+2}\, dt$.

Soln: Let $w = t+2 \implies \frac{dw}{dt} = 1$. Since w is a function of t, we also have

$$t = -1 \implies w(-1) = -1+2 = 1$$
$$t = e-2 \implies w(e-2) = e-2+2 = e.$$

The integral becomes

$$\begin{aligned}\int_{-1}^{e-2} \frac{1}{t+2}\,dt &= \int_1^e \frac{1}{w}\,dw \\ &= \ln|w|\Big|_1^e \\ &= \ln|e| - \ln|1| \\ &= 1.\end{aligned}$$

Trick: Linear substitutions

Recall: A **linear** function is a function of the form $f(x) = mx + b$. Substitution can be used to integrate functions that have a linear part that is in a denominator or nested under a radical sign.

Example (§7.1 #77)

$\int x^2\sqrt{x-2}\,dx$

$\star$ See §7.1 Example 13 and #73-80. See also Example 12 for a more complicated example.

Soln: Let $w = x - 2 \implies dw = dx$ *and* $x = w + 2$.

$$\begin{aligned}
\int x^2\sqrt{x-2}\,dx &= \int (w+2)^2\sqrt{w}\,dw \\
&= \int (w^2+4w+4)\sqrt{w}\,dw \\
&= \int \left(w^{\frac{5}{2}} + 4w^{\frac{3}{2}} + 4w^{\frac{1}{2}}\right)\,dw \\
&= \frac{2}{7}w^{\frac{7}{2}} + \frac{8}{5}w^{\frac{5}{2}} + \frac{8}{3}w^{\frac{3}{2}} + C
\end{aligned}$$

1 Intro to integration techniques

§6.1 Antiderivatives graphically and numerically

§6.2 Constructing antiderivatives analytically

§7.1 Integration by substitution

§7.2 Integration by parts

- Reverse product rule
- Choosing u and dv
- Exceptions and tricks

Reverse product rule

Here's the idea. Recall, the product rule is used to differentiate products of functions, call those functions u and v (instead of the usual f and g):

$$u'(x)v(x) + u(x)v'(x) = \frac{d}{dx}\left(u(x)v(x)\right)$$

Rewrite the derivatives $u'(x)$, $v'(x)$ in **Leibniz notation**. Then integrate both sides:

$$\int \frac{du}{\cancel{dx}} v(x)\, \cancel{dx} + \int u(x) \frac{dv}{\cancel{dx}}\, \cancel{dx} = u(v)v(x) + C$$

Now move the first integral to the other side to get

$$\int u\, dv = uv - \int v\, du + C.$$

This technique is called **integration by parts**, or **IBP**.

Notes:

- Pneumonic device: "ultraviolet voodoo".
- The "$+C$" can go away, since another one will come out of the integral $\int v\, du$.

- Keep in mind u, v are still functions of x; du and dv are functions of x multiplied by dx.
- The new integral that appears should be "easier" than the original integral.
- The trick is in figuring out which function in the integrand is u, and which function is dv.

Example (§7.2 #10)

$\int x^3 \ln x \; dx$

Soln: Put $u = \ln x$ and $dv = x^3 \; dx$. Then we can set $v = \frac{x^4}{4}$, since it has the derivative we want, $\frac{dv}{dx} = x^3$, so that $dv = x^3 \; dx$. We also have $\frac{du}{dx} = \frac{1}{x} \implies du = \frac{1}{x} \; dx$.

Plug u, v, du, and dv into the ultraviolet voodoo formula. Then evaluate the new integral.

$$\int \underbrace{(\ln x)}_{u} \underbrace{x^3\, dx}_{dv} = \underbrace{(\ln x)}_{u} \underbrace{\frac{x^4}{4}}_{v} - \int \underbrace{\frac{x^4}{4}}_{v} \underbrace{\frac{1}{x}\, dx}_{du}$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3\, dx$$

$$= \frac{x^4}{4} \ln x - \frac{x^4}{16} + C$$

Choosing u and dv

How do we know what to choose for u and what to choose for dv? Here's what we want:

- The choice for dv should make v something easy to find.
- We don't want v to be more complicated than $\frac{dv}{dx}$.
- We don't want $\frac{du}{dx}$ to be more complicated than u.

There's also a pneumonic device!

$$\text{Log PoET}$$
$$u \longleftrightarrow dv$$

- Log $=$ logarithmic functions
- Po $=$ polynomial functions (constants, including 1, count as polynomials)
- E $=$ exponential functions (e^x, 2^x, etc.)
- T $=$ trigonometric functions

Exceptions and tricks

Example (§7.2 #13)

$\int \sin^2 x \, dx$

Soln: Let $u = \sin x$ and $dv = \sin x \, dx$. Then $du = \cos x \, dx$ and $v = -\cos x$. We have

$$\int \sin^2 x \, dx = (\sin x)(-\cos x) - \int (-\cos x)(\cos x \, dx)$$
$$= -\sin x \cos x + \int \cos^2 x \, dx.$$

Use the trig identity $\sin^2 x + \cos^2 x = 1$ to replace the $\cos^2 x$.

$$\begin{aligned}\int \sin^2 x \ dx &= -\sin x \cos x + \int (1 - \sin^2 x) \ dx \\ &= -\sin x \cos x + \int 1 \ dx - \int \sin^2 x \ dx.\end{aligned}$$

There is a $\int \sin^2 x \ dx$ on both sides of the equation, so solve for it!

$$\begin{aligned}2\int \sin^2 x \ dx &= -\sin x \cos x + \int 1 \ dx \\ \implies \int \sin^2 x \ dx &= \frac{1}{2}\left(-\sin x \cos x + \int 1 \ dx\right) \\ &= \frac{1}{2}(-\sin x \cos x + x) + C\end{aligned}$$

$\star$ Check that $\frac{d}{dx}\left(\frac{1}{2}(-\sin x \cos x + x)\right) = \sin^2 x$!

Other tricks:

- The same "solve for the integral" trick can be used to integrate $e^{ax}\sin(bx)$ or $e^{ax}\cos(bx)$.

  $\star$ See Example 7 in §7.2.
- IBP can be used to integrate $\ln x$ by writing $\ln x = 1 \cdot \ln x$, then letting $u = \ln x$ and $dv = 1 \cdot dx$ (and similarly to integrate $\arctan x$).

  $\star$ See Example 3.
- Sometimes it's necessary to do IBP more than once to get a complete answer.

  $\star$ See Example 5.

Week 2 overview

2 Advanced integration techniques

§7.3 Tables of integrals

- The table
- Products of powers of sines and cosines
- Long division of polynomials

§7.4 Algebraic identities and trigonometric substitutions

- Partial fractions: the idea
- Common denominator backwards
- How to integrate
- Irreducible quadratic forms
- Completing the square
- Trig substitutions
- Example: invoking the triangle
- Another example

Application problem(s)

- (1) *§7.3 #44 (Stewart)*

2 Advanced integration techniques

§7.3 Tables of integrals

- The table
- Products of powers of sines and cosines
- Long division of polynomials

§7.4 Algebraic identities and trigonometric substitutions

Application problem(s)

The table

A short table of integrals is at the back of the textbook. You can use it to help solve more complicated integrals. A *very* comprehensive list of complicated integrals is given at `integral-table.com`. There is also a ~printable version~, typeset using LaTeX!

Note: The table can't be used unless the integral is in the right form. Here are some techniques to use:

- Substitution. Linear substitutions are especially helpful.
- Trig identities. E.g., $\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx$.
- Log algebra. See the rules of natural logarithms in the front cover of the text.

  $\star$ See §7.3 Example 10.
- Complete the square.

  $\star$ See Example 8.

Products of powers of sines and cosines

Example

$\int \sin^2 x \cos^2 x \; dx$

Guidelines for $\int \sin^m x \cos^n x \; dx$, with $m, n > 0$:

- If both m, n are even, use $\sin^2 x + \cos^2 x = 1$ to convert the integral into either all sines or all cosines.
- If one or both of m, n is odd, then use a substitution $w = \cos x$ or $w = \sin x$ for the other function. Then again use $\sin^2 x + \cos^2 x = 1$ to convert as much as possible into the function that appears an odd number of times.

Soln:

$$\begin{aligned}\int \sin^2 x \cos^2 x \; dx &= \int \sin^2 x(1 - \sin^2 x) \; dx \\ &= \int \sin^2 x \; dx - \int \sin^4 x \; dx\end{aligned}$$

Now use the identity from the table

$$\int \sin^n x \; dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \; dx$$

applied with $n = 2$ and $n = 4$. Note that $\sin^0 x = 1$.

$$\underbrace{\left(-\frac{1}{2}\sin x\cos x+\frac{1}{2}\int dx\right)}_{\int \sin^2 x\ dx}$$

$$-\underbrace{\left(-\frac{1}{4}\sin^3 x\cos x+\frac{3}{4}\int \sin^2 x\ dx\right)}_{\int \sin^4 x\ dx}$$

$$=-\frac{1}{2}\sin x\cos x+\frac{1}{2}x+\frac{1}{4}\sin^3 x\cos x$$

$$-\frac{3}{4}\left(-\frac{1}{2}\sin x\cos x+\frac{1}{2}\int dx\right)$$

$$=-\frac{1}{2}\sin x\cos x+\frac{1}{2}x+\frac{1}{4}\sin^3 x\cos x$$

$$+\frac{3}{8}\sin x\cos x-\frac{3}{8}x+C$$

Long division of polynomials

Recall, a **rational function** is a function of the form

$$f(x) = \frac{\text{polynomial}}{\text{another polynomial}}.$$

The **degree** of a polynomial is the highest power that appears.

If the degree of the numerator is greater than or equal to the degree of the denominator, then we can use long division to make integration easier.

For now we only consider rational functions where the degree of the denominator is at most 2. The bigger cases will be considered in §7.4.

Example

$\int \frac{x^2+1}{x^2-5x+4}\ dx$

Soln: Use long division to make the degree of the numerator smaller:

$$\begin{array}{r|l} & 1 \\ \hline x^2-5x+4 & x^2 + 1 \\ & -\,x^2+5x-4 \\ \hline & 5x-3 \end{array}$$

⋆ For a refresher on long division of polynomials, ~these notes by Scott Pike at Mesa Community College~ contain examples and practice problems with solutions!

This means

$$\frac{x^2+1}{x^2-5x+4} = 1 + \frac{5x-3}{x^2-5x+4}.$$

Now notice the denominator factors:

$$x^2-5x+4 = (x-4)(x-1)$$

So we use the table identity

$$\int \frac{cx+d}{(x-a)(x-b)} = \frac{1}{a-b}\left((ac+d)\ln|x-a| \right.$$
$$\left. -(bc+d)\ln|x-b|\right) + C,$$

applied with $a=4$, $b=1$, $c=5$, $d=-3$.

We have

$$\begin{aligned}\int \frac{x^2+1}{x^2-5x+4}\,dx &= \int dx + \int \frac{5x-3}{(x-4)(x-1)}\,dx \\ &= x + \frac{1}{4-1}\left(((4)(5)+(-3))\ln|x-4|\right. \\ &\qquad \left.-((1)(5)+(-3))\ln|x-1|\right) + C \\ &= x + \frac{1}{3}(17\ln|x-4| - 2\ln|x-1|) + C.\end{aligned}$$

Partial fractions: the idea

Partial fractions is the reverse method of getting a common denominator.

It is used to integrate rational functions whose numerator has degree smaller than the denominator (this means if the numerator has degree greater than or equal to the degree of the denominator, you must do long division first).

Partial fractions relies on the following important theorem from algebra:

Theorem (Fundamental Theorem of Algebra)

Every polynomial with real coefficients factors into a product of linear and irreducible quadratic forms.

This means that no matter the degree of a polynomial, it will always factor into a product of linear $(mx + b)$ and quadratic (degree 2) pieces.

Example

$$x^3 + 1 = \underbrace{(x+1)}_{\text{linear}}\ \underbrace{(x^2 - x + 1)}_{\text{irreducible quadratic}}$$

Common denominator backwards

Example

Suppose the polynomial $x^2 - 5x + 4 = (x-4)(x-1)$ is in the denominator of a rational function that we wish to write in terms of partial fractions.

$$\frac{1}{x^2 - 5x + 4} = \frac{A}{x-4} + \frac{B}{x-1}$$

Soln: To find A and B, first get a common denominator on the right hand side of the equation. Then set the numerators of both sides equal to each other.

$$\begin{aligned}
\frac{1}{x^2-5x+4} &= \frac{A}{x-4}\left(\frac{x-1}{x-1}\right) + \frac{B}{x-1}\left(\frac{x-4}{x-4}\right) \\
&= \frac{Ax-A}{x^2-5x+4} + \frac{Bx-4B}{x^2-5x+4} \\
&= \frac{(A+B)x+(-A-4B)}{x^2-5x+4} \\
\implies 1 &= (A+B)x+(-A-4B)
\end{aligned}$$

Since the left hand side of the equation doesn't have any xs, we must have $A+B=0$. That means we also must have $-A-4B=1$.

We get a system of equations, 2 equations with 2 unknowns:

$$\begin{aligned} A + B &= 0 \\ -A - 4B &= 1 \end{aligned}$$

Use the first equation to get $A = -B$, then substitute into the second equation:

$$\begin{aligned} -A - 4B &= 1 \\ -(-B) - 4B &= 1 \\ -3B &= 1 \\ \implies B &= -\frac{1}{3} \implies A = \frac{1}{3} \end{aligned}$$

The conclusion is that

$$\begin{aligned} \frac{1}{x^2-5x+4} &= \frac{\frac{1}{3}}{x-4} + \frac{-\frac{1}{3}}{x-1} \\ &= \frac{1}{3(x-4)} - \frac{1}{3(x-1)}. \end{aligned}$$

We can check this by finding a common denominator:

$$
\begin{aligned}
\frac{1}{3(x-4)} - \frac{1}{3(x-1)} &= \frac{1}{3(x-4)}\left(\frac{x-1}{x-1}\right) \\
&\quad - \frac{1}{3(x-1)}\left(\frac{x-4}{x-4}\right) \\
&= \frac{\cancel{x} - 1 - (\cancel{x} - 4)}{3(x^2 - 5x + 4)} \\
&= \frac{\cancel{3}}{\cancel{3}(x^2 - 5x + 4)} \\
&= \frac{1}{x^2 - 5x + 4}
\end{aligned}
$$

How to integrate

Why does this work for integration?

We have

$$\int \frac{1}{x^2 - 5x + 4} = \frac{1}{3} \int \frac{1}{x-4}\, dx - \frac{1}{3} \int \frac{1}{x-1}\, dx.$$

On the right hand side, use $w = x - 4$ on the first integral and $w = x - 1$ on the second integral to get

$$\int \frac{1}{x^2 - 5x + 4} = \frac{1}{3}\ln|x-4| - \frac{1}{3}\ln|x-1| + C,$$

which can be verified by the table identity $(a \neq b)$

$$\int \frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left(\ln|x-a| - \ln|x-b|\right) + C$$

with $a = 4$ and $b = 1$.

Irreducible quadratic forms

When the denominator of a rational function has an irreducible quadratic factor, the method of partial fractions changes. We write

$$\frac{1}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$$

(note the captial and small letters are distinct).

The reason is that we want the largest possible degree numerator for each term of the right hand side.

Example (§7.4 #4)

$\int \frac{2y}{y^3-y^2+y-1}\, dy$

Here, the numerator does not have a 1, but the method is the same. To factor the denominator, use a site like ~MathPapa's factoring polynomials calculator~ to get

$$y^3 - y^2 + y - 1 = (y-1)(y^2+1).$$

Partial fractions the integrand to get

$$\begin{aligned}\frac{2y}{y^3-y^2+y-1} &= \frac{A}{y-1}+\frac{By+C}{y^2+1}\\ &= \frac{A}{y-1}\left(\frac{y^2+1}{y^2+1}\right)+\frac{By+C}{y^2+1}\left(\frac{y-1}{y-1}\right)\\ &= \frac{(Ay^2+A)+(By^2+Cy-By-C)}{y^3-y^2+y-1}\\ \implies 2y &= (A+B)y^2+(-B+C)y+(A-C)\end{aligned}$$

The system of equations is

$$\begin{aligned} A + B &= 0 \\ -B + C &= 2 \\ A - C &= 0. \end{aligned}$$

The solutions are $A = 1$, $B = -1$, $C = 1$.

The integral becomes

$$\begin{aligned}\int \frac{2y}{y^3 - y^2 + y - 1}\, dy &= \int \frac{1}{y-1}\, dy - \int \frac{y}{y^2+1}\, dy \\ &\quad + \int \frac{1}{y^2+1}\, dy \\ &= \ln|y-1| - \frac{1}{2}\ln(y^2+1) \\ &\quad + \arctan y + C\end{aligned}$$

where in the second integral, $w = y^2 + 1$ is used.

If a denominator has repeated factors then the method of partial fractions must be modified again.

Example

$$\frac{1}{(x-1)^3(x^2+1)^2} = \underbrace{\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}}_{\text{one term for each } x-1}$$

$$+ \underbrace{\frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}}_{\text{one term for each } x^2+1}$$

$\star$ See §7.4 Example 3.

Completing the square

Sometimes integrating an irreducible quadratic in the denominator does not immediately lead to the arctan function.

Complete the square rewrites a quadratic polynomial as follows:

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 \\ &= \left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 \end{aligned}$$

- Notice there is no coefficient on the x^2 term.

Example (§7.4 #30)

$\int \frac{dx}{x^2+2x+2}$

Soln: Complete the square in the denominator:

$$\begin{aligned} x^2 + 2x + 2 &= x^2 + 2x + 1^2 + 2 - 1^2 \\ &= (x+1)^2 + 1 \end{aligned}$$

Now let $w = x + 1$ to finish solving the integral.

Trig substitutions

Guidelines:

- If the integrand contains $\sqrt{a^2 - x^2}$ for some constant a, then put

$$x = a \sin\theta \implies dx = a\cos\theta \; d\theta.$$

This implies $\theta = \arcsin\frac{x}{a}$ on the interval $-a \leq x \leq a$ if we assume $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. We get

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta.$$

- If the integrand contains $x^2 + a^2$ or $\sqrt{x^2 + a^2}$, then put
$$x = a \tan\theta \implies dx = a\sec^2\theta \; d\theta.$$
Assume $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ so that $\theta = \arctan\frac{x}{a}$ for all x. We get
$$x^2 + a^2 = a^2\tan^2\theta + a^2 = a^2\sec^2\theta.$$

$\star$ The substitution $x = a\sec\theta$ for $x^2 - a^2$ also exists, but we will not cover those examples in this course.

Example: invoking the triangle

Example (§7.4 #58)

$\int \frac{1}{x\sqrt{9-4x^2}}\ dx$

Soln: The expression under the radical sign is not of the form $a^2 - x^2$ so we first need to rewrite it:

$$\sqrt{9-4x^2} = \sqrt{4\left(\frac{9}{4} - \frac{4}{4}x^2\right)}$$

$$= 2\sqrt{\left(\frac{3}{2}\right)^2 - x^2}$$

$$\implies \int \frac{1}{x\sqrt{9-4x^2}}\,dx = \frac{1}{2}\int \frac{1}{x\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}\,dx$$

Put $x = \frac{3}{2}\sin\theta$. Then $dx = \frac{3}{2}\cos\theta\ d\theta$. The integral becomes

$$\frac{1}{2}\int \frac{\frac{3}{2}\cos\theta\ d\theta}{\frac{3}{2}\sin\theta\left(\frac{3}{2}\cos\theta\right)} = \frac{1}{3}\int \frac{d\theta}{\sin\theta}.$$

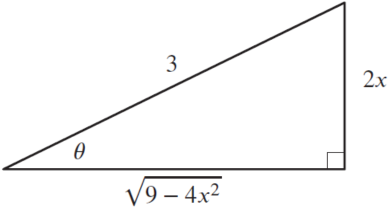
From the table, we have

$$\frac{1}{3}\int \frac{d\theta}{\sin\theta} = \frac{1}{6}\ln\left|\frac{\cos\theta+1}{\cos\theta-1}\right| + C.$$

But we aren't done! The answer should be written in terms of x, not θ.

How do we know $\cos\theta$ in terms of x, when we're only given $x = \frac{3}{2}\sin\theta$?

The answer is to **invoke the triangle**. We want a right triangle with angle θ such that $\sin\theta = \frac{2x}{3}$.



Graphic: Triangle with $\sin\theta = \frac{2x}{3}$.

From the picture, we conclude $\cos\theta = \frac{\sqrt{9-4x^2}}{3}$. Plug that into the final equation:

$$\frac{1}{6}\ln\left|\frac{\cos\theta+1}{\cos\theta-1}\right| + C = \frac{1}{6}\ln\left|\frac{\frac{\sqrt{9-4x^2}}{3}+1}{\frac{\sqrt{9-4x^2}}{3}-1}\right| + C.$$

Another example

Example (§7.4 #63)

$$\int \frac{x^2}{(1+9x^2)^{\frac{3}{2}}}\, dx$$

Soln: The $\frac{3}{2}$ in the exponent is really a cube over a radical sign. Rewrite the denominator:

$$\begin{aligned}(1+9x^2)^{\frac{3}{2}} &= \left(\sqrt{1+9x^2}\right)^3\\ &= \left(\sqrt{9\left(\frac{1}{9}+x^2\right)}\right)^3\\ &= 27\left(\sqrt{\left(\frac{1}{3}\right)^2+x^2}\right)^3\end{aligned}$$

To integrate, put $x = \frac{1}{3}\tan\theta$. Then $dx = \frac{1}{3}\sec^2\theta\ d\theta$.

$$\int \frac{x^2}{27\left(\sqrt{\left(\frac{1}{3}\right)^2 + x^2}\right)^3}\,dx = \int \frac{\left(\frac{1}{3}\tan\theta\right)^2}{27(\frac{1}{3}\sec\theta)^3}\left(\frac{1}{3}\sec^2\theta\right)d\theta$$

$$= \frac{1}{27}\int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

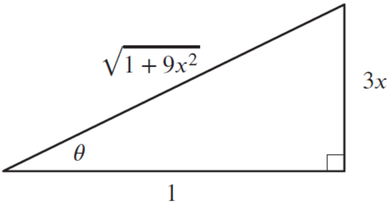
$$= \frac{1}{27}\int \frac{\frac{\sin^2\theta}{\cos^{\cancel{2}}\theta}}{\frac{1}{\cancel{\cos\theta}}}\,d\theta$$

$$= \frac{1}{27}\int \frac{\sin^2\theta}{\cos\theta}\,d\theta$$

Use the identity $\sin^2\theta + \cos^2\theta = 1$ and then the table for $\int \frac{1}{\cos\theta}\, d\theta$.

$$\begin{aligned}\frac{1}{27}\int \frac{\sin^2\theta}{\cos\theta}\, d\theta &= \frac{1}{27}\int \frac{1-\cos^2\theta}{\cos\theta}\, d\theta \\ &= \frac{1}{27}\left(\int \frac{1}{\cos\theta}\, d\theta - \int \cos\theta\, d\theta\right) \\ &= \frac{1}{27}\left(\frac{1}{2}\ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right| - \sin\theta\right) + C\end{aligned}$$

Invoke the triangle to write the answer in terms of x. We have $x = \frac{1}{3}\tan\theta$ so we want $\tan\theta = 3x = \frac{3x}{1}$.



Graphic: Triangle with $\tan\theta = \frac{3x}{1}$.

From the Pythagorean theorem and the picture, we conclude $\sin\theta = \frac{3x}{\sqrt{1+9x^2}}$.

$$\frac{1}{27}\left(\frac{1}{2}\ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right| - \sin\theta\right) + C$$

$$= \frac{1}{54}\left(\ln\left|\frac{\frac{3x}{\sqrt{1+9x^2}}+1}{\frac{3x}{\sqrt{1+9x^2}}-1}\right| - \frac{3x}{\sqrt{1+9x^2}}\right) + C$$

2 Advanced integration techniques

§7.3 Tables of integrals

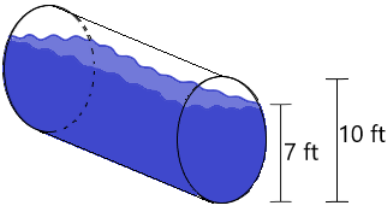
§7.4 Algebraic identities and trigonometric substitutions

Application problem(s)

- (1) *§7.3 #44 (Stewart)*

(1) *§7.3 #44 (Stewart)*

A water storage tank has the shape of a cylinder with diameter 10 ft. It is mounted so that the circular cross-section is vertical. If the depth of the water is 7 ft, what percentage of the total capacity is being used?



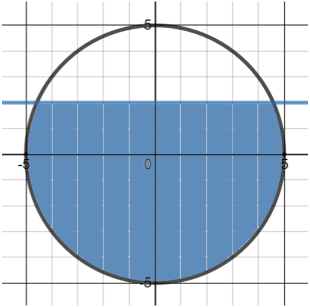
Graphic: Water storage tank filled to 7 ft.

Hint: Try to visualize the problem in two dimensions.

Soln: We can look at one circular cross-section at a time. The percentage capacity used to fill the circle to 7 ft will equal the percentage capacity used in the cylindrical tank.

The problem then translates into a problem about the area between two curves. One is a circle, corresponding to the tank. The other is a line, corresponding to the top of the water.

The circle has diameter 10 ft, so use the circle $x^2 + y^2 = 25$. It is centered on the origin. The line representing the top of the water is $y = 2$, since it is 7 units above the bottom of the circle.



Graphic: Vertical cross-section of the cylindrical tank.

We run into a problem trying to find the shaded area. The bottom portion of the circle includes the bottom half, and part of the top half. It's not given by an explicit equation (i.e., $y =$ an expression in only xs).

However, the top portion is a subset of the top half of the circle, so we can use the equation $y = \sqrt{25 - x^2}$.

We will need to find the area between the top portion of the circle and the line $y = 2$, then take the difference from the total area of the circle (25π) to get the area of the shaded region.

To find the area, we set up an integral. We first need to find the bounds by setting the equations of the curves equal to each other:

$$\begin{aligned} y = \sqrt{25 - x^2} &= 2 \\ 25 - x^2 &= 4 \\ \implies\ 21 = x^2 &\implies\ x = \pm\sqrt{21} \end{aligned}$$

Now we evaluate the integral

$$\int_{\sqrt{21}}^{\sqrt{21}} \sqrt{25 - x^2} - 2\; dx = 2\int_0^{\sqrt{21}} \sqrt{25 - x^2} - 2\; dx,$$

since the integrand is an even function (symmetric about the y-axis).

To evaluate the integral, use linearity and then the trig substitution $x = 5\sin\theta$. Note that since we are doing a substitution for a definite integral, we must change the bounds. We have $\theta = \arcsin\frac{x}{5}$. The integral becomes

$$2\int_0^{\sqrt{21}} \sqrt{25 - x^2}\ dx - 2\int_0^{\sqrt{21}} 2\ dx$$

$$= 2\int_{\arcsin\frac{0}{5}}^{\arcsin\frac{\sqrt{21}}{5}} (5\cos\theta)(5\cos\theta\ d\theta) - 4x\Big|_0^{\sqrt{21}}$$

$$= 50\int_0^{\arcsin\frac{\sqrt{21}}{5}} \cos^2\theta\ d\theta - 4\sqrt{21}$$

$(\arcsin 0 = 0)$.

Now use the table:

$$50\int_0^{\arcsin\frac{\sqrt{21}}{5}} \cos^2\theta\ d\theta - 4\sqrt{21}$$

$$= 50\left(\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right)\Bigg|_0^{\arcsin\frac{\sqrt{21}}{5}} - 4\sqrt{21}$$

$$= 50\left(\frac{\sin\left(2\arcsin\frac{\sqrt{21}}{5}\right)}{4} + \frac{\arcsin\frac{\sqrt{21}}{5}}{2}\right) - 4\sqrt{21}$$

Subtracting from the area of the entire circle, we get the area of the shaded region. Then we divide by the area of the entire circle to get a percentage:

$$\frac{25\pi - \left(50\left(\frac{\sin\left(2\arcsin\frac{\sqrt{21}}{5}\right)}{4} + \frac{\arcsin\frac{\sqrt{21}}{5}}{2}\right) - 4\sqrt{21}\right)}{25\pi}$$
$$\approx 0.748,$$

or 74.8%.

Week 3 overview

3 Applications of integration

§8.1 Areas and volumes

- Recall: Riemann sums
- Areas and volumes of shapes
- Recall: trig substitution

§8.2 Applications to geometry

- Volumes by discs
- Volumes by rings
- Volumes of regions of known cross-section
- Arc length

§7.6 Improper integrals

- Infinite bounds
- Singularities in the integrand

Application problem(s)

- (1) *§8.2 #30 (Stewart)*
- (2) *§8.2 #40 (Lial, et al.)*
- (3) *§7.6 #50*

Week 3 overview (cont.)

3 Applications of integration

§8.1 Areas and volumes

- Recall: Riemann sums
- Areas and volumes of shapes
- Recall: trig substitution

§8.2 Applications to geometry

§7.6 Improper integrals

Application problem(s)

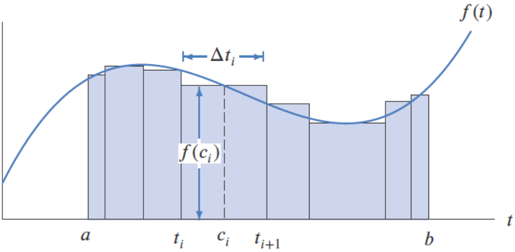
Recall: Riemann sums

A general Riemann sum for f on the interval $[a, b]$ is of the form

$$\int_a^b f(t)\, dt \approx \sum_{i=1}^{n} f(c_i)\Delta t_i.$$

Notes:

- The number of rectangles under the curve is n. As the number of rectangles gets bigger, we get a better approximation of the integral. We get the exact answer when we take $\lim_{n\to\infty}$.
- $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$



Graphic: Area under the curve approximation with a general Riemann sum.

- For $i = 1, \ldots, n$ the endpoints of Rectangle #i are t_{i-1} and t_i. The thickness of Rectangle #i is $\Delta t_i = t_i - t_{i-1}$, and c_i is a number between t_{i-1} and t_i.
- The Δt_is don't have to all be the same size.
- The $f(c_i)$s give the height of the rectangles, and don't have to all be the same height.

$\star$ For refresher on Riemann sums, see §5.1-5.2.

Areas and volumes of shapes

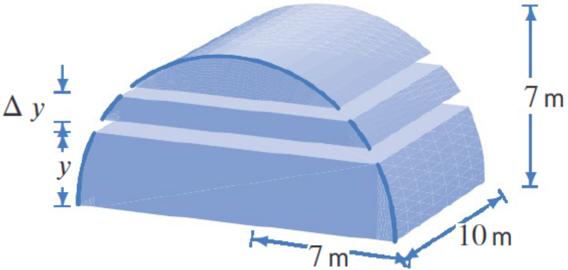
$\star$ We will focus on volumes. See §8.1 Examples 1-2 for an alternate method of finding the area of a triangle and a semicircle (that uses Riemann sums!).

Riemann sums for the area under the curve work because we are adding up areas of rectangles, and the area of a rectangle has an easy formula.

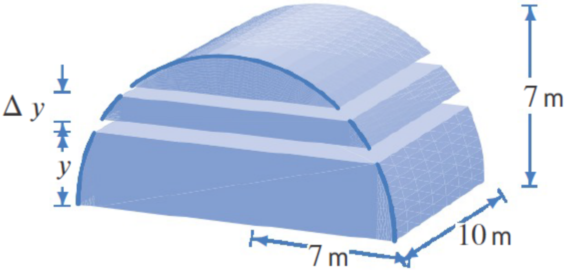
We can apply the same idea to approximating the volume of solid shapes.

Example (§8.1 #17)

Write a Riemann sum and then a definite integral representing the volume of the region, using the slice shown. Evaluate the integral exactly.



Graphic: Solid semicircle volume approximated using strips.

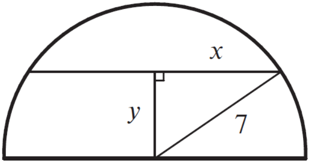


Graphic: Solid semicircle volume approximated using strips.

Soln: The solid is divided into approximate boxes. The volume of a box is length $\times$ width $\times$ height.

Each box has height Δy. The width is given as 10m.

To find the length of each box, we need to look at a cross section of the solid.



Graphic: Semicircle cross-section of solid.

We want to find the length $2x$. The radius of the solid is 7 m, so gives the hypotenuse of the right triangle shown. The Pythagorean theorem says

$$2x = 2\sqrt{7^2 - y^2}.$$

The number of boxes used is arbitrary, so we suppress the is in the notation. The Riemann sum is

$$\sum \underbrace{\left(2\sqrt{7^2 - y^2}\right)}_{\text{length}} \underbrace{(10)}_{\text{width}} \underbrace{(\Delta y)}_{\text{height}}.$$

To write the definite integral, look at the range on y. The minimum value is 0 and the maximum value is 7. The definite integral is

$$\lim_{\Delta y \to 0} 20 \sum \sqrt{7^2 - y^2}\, \Delta y = 20 \int_0^7 \sqrt{7^2 - y^2}\, dy.$$

Recall: trig substitution

Now we evaluate the integral using a trig substitution. Put $y = 7\sin\theta$. Then $dy = 7\cos\theta\ d\theta$, and $\theta = \arcsin\frac{y}{7}$.

$$20\int_0^7 \sqrt{7^2 - y^2}\ dy = 20\int_{\arcsin\frac{0}{7}}^{\arcsin\frac{7}{7}} 7\cos\theta\,(7\cos\theta\ d\theta)$$

$$= 980\int_0^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

Note that $\arcsin 1 = \frac{\pi}{2}$.

Use the table of integrals:

$$\begin{aligned} 980\int_0^{\frac{\pi}{2}} \cos^2\theta \; d\theta &= 980\left(\frac{1}{2}\cos\theta\sin\theta\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta\right) \\ &= 490\left(\cancel{\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)} + \left(\frac{\pi}{2} - \cancel{0}\right)\right) \\ &= 245\pi \text{ m}^3. \end{aligned}$$

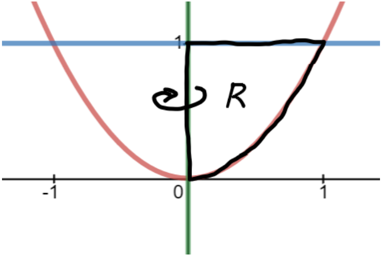
$\star$ Check that this is consistent with the formula for the area of a half cylinder!

3 Applications of integration

Volumes by discs

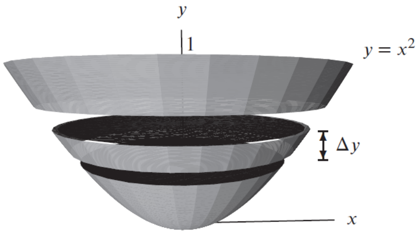
Example (§8.2 #44)

Consider the region R bounded by the curve $y = x^2$, the line $y = 1$, and the y-axis, with $x \geq 0$. Find the volume of the solid obtained by rotating R around the y-axis.



Graphic: Region R of the parabola to be rotated about the y-axis.

Soln: The solid can be divided into approximate discs (flat cylinders) whose volumes we can sum.

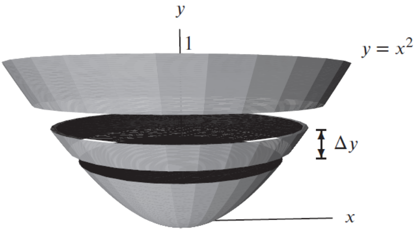


Graphic: Parabola rotated about the y-axis.

The volume of a cylinder is

(area of a circle) $\times$ (thickness of the cylinder).

The thickness of one of the discs is Δy. We need to find the radius.



Graphic: Parabola rotated about the y-axis.

The equation $y = x^2$ gives the height of one of the discs. To find the length we need the x-coordinate of the boundary of the disc. We find it by solving for x:

$$y = x^2 \implies x = \sqrt{y}$$

The volume of one disc is $\pi \left(\sqrt{y}\right)^2 \Delta y$.

The total volume is

$$\begin{aligned}\pi \lim_{\Delta y \to 0} \sum y \, \Delta y &= \pi \int_0^1 y \, dy \\ &= \pi \frac{y^2}{2}\Big|_0^1 \\ &= \frac{1}{2}\pi.\end{aligned}$$

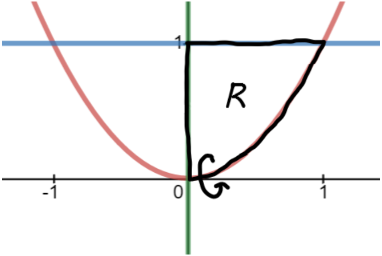
The bounds on the integral come from the fact that R is bounded by the x-axis ($y = 0$) and the line $y = 1$.

⋆ See also §8.2 Examples 1-2.

Volumes by rings

Example (§8.2 #45)

R from the previous example is now rotated around the x-axis. Find the volume of the solid.

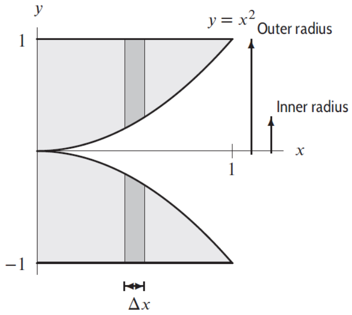


Graphic: Region R to be rotated about the x-axis.

Soln: The vertical cross sections of the solid are rings. The volume of a ring is

$$\pi\left((\text{outer radius})^2 - (\text{inner radius})^2\right) \times (\text{thickness}).$$

The thickness of a ring is Δx. We can find the inner and outer radii by looking at a flat cross section of the solid.



Graphic: Cross-section of the solid.

The inner radius is the height of the curve, $y = x^2$. The outer radius is given by the line $y = 1$, so is 1.

To find the bounds for the integral we look at the min and max values of x for R.

R is bounded by the y-axis so the lower bound is $x = 0$.

On the right R is bounded where the line $y = 1$ meets the curve $y = x^2$. Setting $1 = x^2$ gives $x = \pm 1$ but we take the positive value, since $x \geq 0$.

The volume is

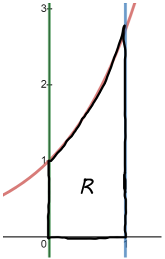
$$\begin{aligned}
\pi \lim_{\Delta x \to 0} \sum \left(1^2 - (x^2)^2\right) \Delta x &= \pi \int_0^1 \left(1 - x^4\right)\, dx \\
&= \pi \left(x - \frac{x^5}{5} \right) \Bigg|_0^1 \\
&= \pi \left(1 - \frac{1}{5} \right) \\
&= \frac{4}{5}\pi.
\end{aligned}$$

⋆ See also §8.2 Example 3.

Volumes of regions of known cross-section

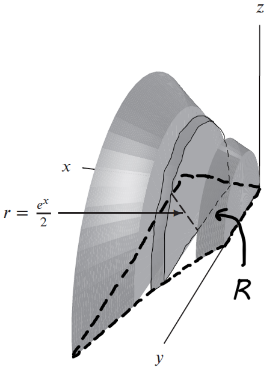
Example (§8.2 #54)

Consider the region R bounded by the curve $y = e^x$, the x-axis, and the lines $x = 0$ and $x = 1$. Find the volume of the solid whose cross-sections perpendicular to the x-axis are semicircles.



Graphic: Region R cut out by $y = e^x$, $x = 1$, and the axes.

The semicircles come out of the plane (the z-axis), with the long edges parallel to the y-axis. In the picture, the xy-plane is rotated upside-down.



Graphic: Semicircles on top of the region R.

The radius of a semicircle is half the height of the curve $y = e^x$, so is $\frac{e^x}{2}$.

The volume of one semicircle wedge is

$$\frac{1}{2}\pi(\text{radius})^2(\text{thickness}).$$

The thickness of one semicircle is Δx. The volume is

$$\frac{1}{2}\pi \lim_{\Delta x \to 0} \sum \left(\frac{e^x}{2}\right)^2 \Delta x = \frac{\pi}{2}\int_0^1 \frac{e^{2x}}{4}\, dx.$$

To integrate, use $w = 2x$, so that $dx = \frac{1}{2}\, dw$, $w(0) = 0$, and $w(1) = 2$.

$$\frac{\pi}{8}\int_0^2 \frac{e^w}{2}\, dw = \frac{\pi}{16} e^w \Big|_0^2 = \frac{\pi}{16}\left(e^2 - 1\right)$$

$\star$ See also §8.2 Example 4.

Arc length

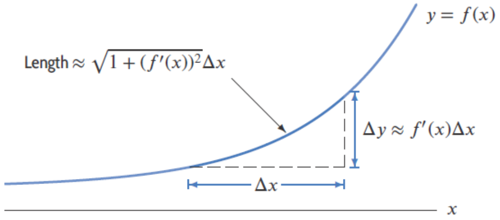
Q: How do we measure the length of a piece of string when we can't stretch it straight?

Ans: Divide it into tiny lengths, so that each tiny length is the hypotenuse of a right triangle with sides Δx and Δy.

From Calc I, we have $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$, so we can write $\Delta y \approx \frac{dy}{dx} \Delta x$.

Suppose the piece of string is given by the curve $y = f(x)$. Use the Pythagorean theorem to find a tiny length:

$$\begin{aligned}
\text{tiny length} &\approx \sqrt{(\Delta x)^2 + (\Delta y)^2} \\
&\approx \sqrt{(\Delta x)^2 + \left(\frac{dy}{dx}\Delta x\right)^2} \\
&= \sqrt{(\Delta x)^2 + \left(\frac{dy}{dx}\right)^2 (\Delta x)^2} \\
&= \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)(\Delta x)^2} \\
&= \sqrt{1 + f'(x)^2}\,\Delta x
\end{aligned}$$



Graphic: Pythagoren theorem used to approximate a small length of the curve.

The arc length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$\lim_{\Delta x \to 0} \sum \sqrt{1 + f'(x)^2}\Delta x = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

$\star$ §8.2 Examples 6-7 use the arc length formula for **parametric** curves, but we will not cover those in this course.

3 Applications of integration

§8.1 Areas and volumes

§8.2 Applications to geometry

§7.6 Improper integrals

- Infinite bounds
- Singularities in the integrand

Application problem(s)

Infinite bounds

There are two types of improper integrals:

1. Integrals where one or both of the bounds is infinity.
2. Integrals over intervals where the function $f(x)$ reaches infinity.

Type 1: Infinite bounds

Example

Find the area under the curve $f(x) = \frac{1}{x^2}$ for $x \geq 1$.

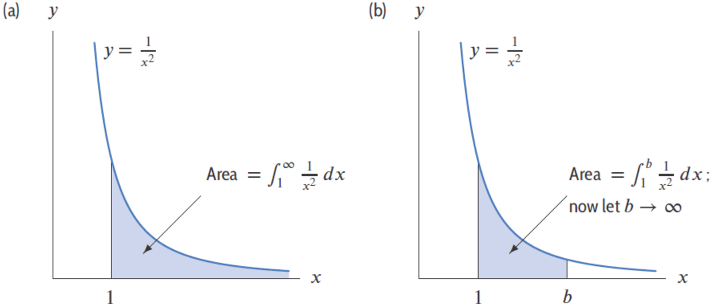
The area is given by the improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx.$$

It doesn't seem like such an area should be finite, since we are considering a region with an infinite length. However, we will show that the area is, in fact, finite.

Soln: To evaluate, replace ∞ with a placeholder variable b. Then

$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \int_1^\infty \frac{1}{x^2}\,dx.$$



Graphic: Area representation of an improper integral with infinite bounds.

Evaluate the integral on the lefthand side and take the limit as $b \to \infty$.

$$\begin{aligned}\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\, dx &= \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b \\ &= \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) \\ &= 1,\end{aligned}$$

since $\lim_{b\to\infty} \frac{1}{b} = 0$.

We conclude the area under the curve is 1. Since we get a finite number when we take the limit, the area is finite and we say the integral $\int_1^\infty \frac{1}{x^2}\, dx$ **converges**.

$\star$ For an example of an improper integral that **diverges**, see §7.6 Examples 1 and 3.

If both bounds on an improper integral are infinity, we choose any finite number c and write

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{a\to-\infty} \int_a^c f(x)\, dx + \lim_{b\to\infty} \int_c^b f(x)\, dx.$$

Singularities in the integrand

Type 2: Singularities in the integrand

Suppose we wish to evaluate $\int_a^b f(x)\ dx$ but there is a number c between a and b where $\lim_{x\to c} f(x) = \pm\infty$. We say f "blows up", or has a **singularity** at $x = c$.

Since we can write

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx,$$

we assume either $c = a$ or $c = b$.

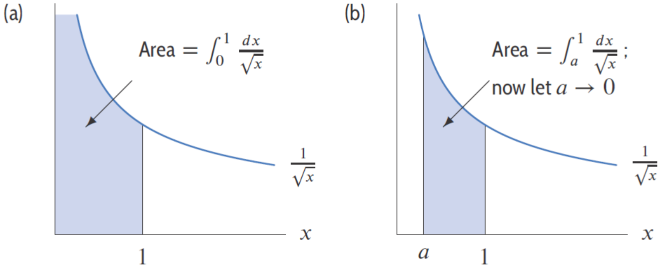
Example

Find the area under the curve $f(x) = \frac{1}{\sqrt{x}}$ between $x = 0$ and $x = 1$.

Soln: Since we can't have a zero in the denominator, f has a singularity at $x = 0$.

We use a placeholder variable $a > 0$, evaluate the integral, and take the limit as a approaches 0 from the right, since $a > 0$.

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} 2x^{\frac{1}{2}}\Big|_a^1$$
$$= \lim_{a\to 0^+} 2 - 2a^{\frac{1}{2}} = 2$$



Graphic: Area representation of an improper integral with singularities in the integrand.

3 Applications of integration

Application problem(s)

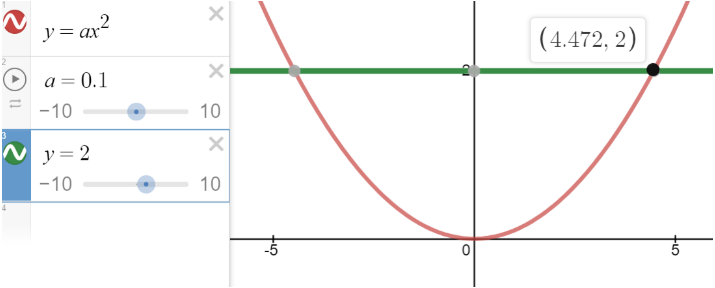
- (1) *§8.2 #30 (Stewart)*
- (2) *§8.2 #40 (Lial, et al.)*
- (3) *§7.6 #50*

(1) *§8.2 #30 (Stewart)*

A group of engineers is building a parabolic satellite dish whose shape will be formed by rotating the curve $y = ax^2$ about the y-axis. If the dish is to have a 10 ft diameter and a maximum depth of 2 ft, find the value of a and the surface area of the dish.

Soln: First graph the parabola. Since the height of the dish must be 2 ft, we intersect the parabola with the line $y = 2$.

~Desmos~ has a slider feature we can use to approximate the value of a. Since the dish is to have diameter 10 ft, its radius should be 5. The value $a = 0.1$ gets us close to intersecting $y = ax^2$ and $y = 2$ at $x = 5$.

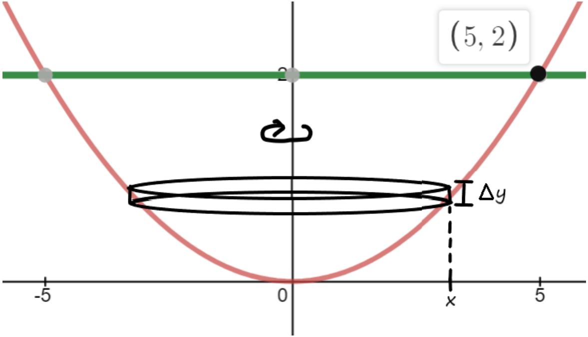


Graphic: $y = ax^2$ intersected with $y = 2$, along with the slider feature for a.

Now we find a exactly using algebra. Since we want $y = x^2$ and $y = 2$ to intersect at $x = 5$, we set the two equations equal to each other with 5 plugged in for x:

$$a(5)^2 = 2 \implies a = \frac{2}{25} = 0.08$$

To find the surface area, we divide the rotated parabola into hollow cylinders and add up the surface area of each cylinder.



Graphic: Parabolic dish divided into hollow cylinders.

Each hollow cylinder has surface area

$$2\pi \times (\text{radius}) \times (\text{thickness}),$$

or the perimeter of a circle times the thickness.

The thickness of a hollow cylinder is Δy. The radius of a hollow cylinder is the x-coordinate, which we get by solving $y = ax^2$ for x; we get $x = \sqrt{\frac{y}{a}}$.

(Even though we already solved for a, it's good practice to leave it as is until the end of the problem.)

We get a Riemann sum and an integral

$$
\begin{aligned}
2\pi \sum \sqrt{\frac{y}{a}}\Delta y &\approx \frac{2\pi}{\sqrt{a}} \int_0^2 \sqrt{y}\, dy \\
&= \frac{2\pi}{\sqrt{a}} \frac{2y^{\frac{3}{2}}}{3}\Bigg|_0^2 \\
&= \frac{2\pi}{\sqrt{\frac{2}{25}}} \frac{2(2)^{\frac{3}{2}}}{3} \\
&\approx 41.9 \text{ ft}^2.
\end{aligned}
$$

(2) §8.2 #40 (*Lial, et al.*)

The flow of blood in an artery of the body is **laminar** (in layers), with the velocity very low near the artery walls and highest in the center of the artery. In this model of blood flow, we calculate the total flow in the artery by thinking of the flow as being made up of many layers of concentric tubes sliding one on the other.

Suppose R is the radius of the artery and r is the radius of a given layer. Then the velocity of blood in that given layer can be shown to be

$$v(r) = k(R^2 - r^2),$$

where k is a constant.

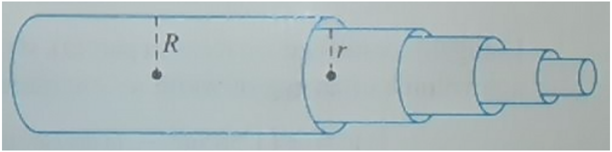
The total flow in the layer is defined to be the product of the velocity and the cross-section area; the cross-section area can be approximated by $dA = 2\pi r\ dr \approx 2\pi r \Delta r$ (since $A = \pi r^2$).

Therefore, the total flow through one layer is

$$F(r) = 2\pi k r(R^2 - r^2)\Delta r.$$

Set up and evaluate a definite integral to find the total flow in the artery.

Q: What does the model look like?
Ans:



Graphic: Laminar model of blood flow through an artery.

Soln: The given foruma $F(r)$ is for the blood flow through one concentric tube. To find the total blood flow, we add up the flow through all the concentric tubes.

$$\sum F(r) = \sum 2\pi k r(R^2 - r^2)\Delta r \approx 2\pi k \int_0^R r(R^2 - r^2)\, dr$$

To integrate, let $w = R^2 - r^2$. Then $-\frac{1}{2}\, dw = r\ dr$. The bounds become $w(R) = R^2 - (R)^2 = 0$ and $w(0) = R^2 - (0)^2 = R^2$.

$$
\begin{aligned}
2\pi k \int_0^R r(R^2 - r^2)\, dr &= \cancel{2}\pi k \left(-\frac{1}{\cancel{2}}\right) \int_{R^2}^0 w \; dw \\
&= -\pi k \frac{w^2}{2}\bigg|_{R^2}^0 \\
&= -\pi k \left(-\frac{(R^2)^2}{2}\right) \\
&= \frac{\pi k R^4}{2}.
\end{aligned}
$$

(3) §7.6 #50

The probability that a light bulb manufactured by a company lasts at least a hundred hours is

$$\int_a^\infty 0.012e^{-0.012t}\,dt.$$

The CEO claims that 90% of the company's light bulbs last at least 1000 hours. Is this statement accurate?

Soln: To answer this question, we find the probability that a light bulb lasts at least 1000 hours. This means we use the given formula with $a = 10 = 1000$ hours plugged in (pay attention to the units!).

Use the substitution $w = -0.012t$. Make sure to change the bounds.

$$
\begin{aligned}
\int_{10}^{\infty} 0.012e^{-0.012t}\,dt &= \cancel{0.012}\int_{-0.12}^{-\infty} \frac{e^w}{-\cancel{0.012}}\,dw \\
&= -\lim_{b\to-\infty}\int_{-0.12}^{b} e^w\,dw \\
&= -\lim_{b\to-\infty} e^w\Big|_{-0.12}^{b} \\
&= -\lim_{b\to-\infty}\left(\overset{0}{\cancel{e^b}} - e^{-0.12}\right) \\
&= e^{-0.12} \approx 0.887
\end{aligned}
$$

This means only 88.7% of the light bulbs last at least 1000 hours. So the company's CEO is lying!

Week 4 overview

4 Intro to infinite series

§7.7 Comparison of improper integrals

- Comparison of improper integrals
- Examples

§9.1 Sequences

- General terms
- Recursively defined sequences
- Convergence of sequences

§9.2 Geometric series

- Series
- Geometric series

§9.3 Convergence of series

- Sequence of partial sums
- The Divergence Test
- The Integral Test
- The p-series Test

Application problems

Week 4 overview (cont.)

- (1) *§9.1 #63*
- (2)

Comparison of improper integrals

Sometimes we can predict whether an improper integral converges before computing it.

There are a few basic forms of integrals whose converge/divergence we know. For other integrals we can use algebraic manipulation to compare them to the basic forms.

The idea is to look at the integrands. Suppose we want to compare $\int_a^b f(x)\ dx$ and $\int_a^b g(x)\ dx$ (where one or both of a, b may be infinity).

Comparisons:

- If $f(x) \leq g(x)$ for $a \leq x \leq b$, then if $\int_a^b g(x)\ dx$ converges, so does $\int_a^b f(x)\ dx$.

  This makes sense because $\int_a^b g(x)\ dx$ is a finite number and $\int_a^b f(x)\ dx$ is a sum of quantities less than that finite number.
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then if $\int_a^b g(x)\ dx$ diverges, then so does $\int_a^b f(x)\ dx$.

  This is because we have a sum of quantities bigger than a sum that goes to infinity.

There are three basic integrals we compare with:

- $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$, diverges for $p \le 1$.
- $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$, diverges for $p \ge 1$.
- $\displaystyle\int_0^\infty e^{-ax}\,dx$ converges for all $a > 0$.

Examples

Example (§7.7 #22)

Decide whether $\int_1^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$ converges.

Soln: Since θ^2 is under a radical sign, we compare it to $\int_1^\infty \frac{d\theta}{\theta}$, which diverges.

If we can show that $\frac{1}{\sqrt{\theta^2+1}} \geq \frac{1}{\theta}$ for $\theta \geq 1$, then it will imply that $\int_1^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$ diverges, too. We have

$$\begin{aligned} \theta^2 + 1 &\leq \theta^2 + \theta^2 \text{ when } \theta \geq 1 \\ \implies \sqrt{\theta^2+1} &\leq \sqrt{2\theta^2} \\ &= \sqrt{2}\theta \text{ because } \theta \geq 1 > 0 \\ \implies \frac{1}{\sqrt{\theta^2+1}} &\geq \frac{1}{\sqrt{2}\theta}. \end{aligned}$$

This is almost what we want, except for the factor of $\frac{1}{\sqrt{2}}$.

Q: Does $\int_1^\infty \frac{d\theta}{\sqrt{2}\theta}$ diverge?

Ans: Yes, since

$$\int_1^\infty \frac{d\theta}{\sqrt{2}\theta} = \frac{1}{\sqrt{2}} \int_1^\infty \frac{d\theta}{\theta} = \frac{1}{\sqrt{2}} \cdot \infty = \infty.$$

We conclude that $\int_1^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$ diverges.

Example (§7.7 #28)

Determine the convergence of $\int_0^\pi \frac{2-\sin\phi}{\phi^2}\,d\phi$.

Soln: When $0 \leq \phi \leq \pi$, we have $0 \leq \sin\phi \leq 1$. This means $1 \leq 2 - \sin\phi$. Thus

$$\frac{1}{\phi^2} \leq \frac{2-\sin\phi}{\phi^2}.$$

The question now is whether $\int_0^\pi \frac{d\phi}{\phi^2}$ diverges. If it does, then so does $\int_0^\pi \frac{2-\sin\phi}{\phi^2}\,d\phi$.

We have

$$\int_0^1 \frac{d\phi}{\phi^2} \leq \int_0^\pi \frac{d\phi}{\phi^2},$$

because we are integrating over a larger area. Since $\int_0^1 \frac{d\phi}{\phi^2}$ diverges, so does $\int_0^\pi \frac{d\phi}{\phi^2}$. Therefore, so does $\int_0^\phi \frac{2-\sin\phi}{\phi^2}\, d\phi$.

4 Intro to infinite series

General terms

A **sequence** $\{s_n\}$ is a (usually infinite) list of numbers.

Example

$$\{s_n\} = \{\underbrace{7}_{s_1}, \underbrace{10}_{s_2}, \underbrace{13}_{s_3}, \underbrace{16}_{s_4}, \underbrace{19}_{s_5}, \dots\}$$

The sequence doesn't need to have a pattern, like the example above does. In either case, we called s_n the **general term**.

When the sequence does have a pattern, it is useful to give it in terms of its general term.

Example

Find the general term for the sequence given in the previous example.

Soln: The general term is $s_n = 7 + 3(n - 1)$. This is because when $n = 1$ we have $s_1 = 7 + 3(1 - 1) = 7$, when $s = 2$ we have $s_2 = 7 + 3(2 - 1) = 10$, when $n = 3$ we have $s_3 = 7 + 3(3 - 1) = 13$, etc.

Example

Write out the first five terms of the sequence $s = \{\underbrace{n + (-1)^n}_{s_n}\}$.

Soln: Plug in values for n:

$$\begin{aligned}
n = 1 &\implies s_1 = 1 + (-1)^1 = 0 \\
n = 2 &\implies s_2 = 2 + (-1)^2 = 3 \\
n = 3 &\implies s_3 = 3 + (-1)^3 = 2 \\
n = 4 &\implies s_4 = 4 + (-1)^4 = 5 \\
n = 5 &\implies s_5 = 5 + (-1)^5 = 4
\end{aligned}$$

Therefore $s = \{0, 3, 2, 5, 4, \dots\}$.

Recursively defined sequences

The **Fibonacci sequence**

$$f = \{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

satisfies

$$\begin{aligned} f_1 = f_2 &= 1 \\ f_3 &= f_1 + f_2 \\ f_4 &= f_2 + f_3 \\ &\vdots \\ f_n &= f_{n-2} + f_{n-1}. \end{aligned}$$

Because each term of f depends on previous terms, the Fibonacci sequence is an example of a **recursively defined sequence**.

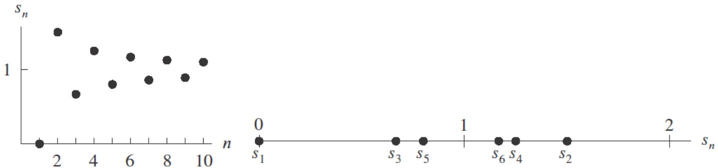
The recursion $f_n = f_{n-2} + f_{n-1}$ requires knowledge of the previous two terms. This means to define the sequence we need two **initial values** $f_1 = 1$ and $f_2 = 1$.

$\star$ Finding a **closed form** general term for a recursively defined sequence, that is, a general term that doesn't depend on previous terms, can be very hard. For example, with some work, it can be shown that the closed form general term of the Fibonacci sequence is an expression in terms of the **Golden Ratio**.

Convergence of sequences

Knowing each term of a sequence requires "plugging in" a value of n to get each term. We can think of a sequence $\{s_n\}$ as a function $y = s(n)$ that takes integers as an input. We can even graph sequences.

On the left the graph is a collection of dots, since the only inputs are the integers. On the right, terms of the sequence are plotted on a number line.



Graphic: The sequence $\{s_n\} = \left\{1 + \frac{(-1)^n}{n}\right\}$.

In both cases the terms of the sequence $\{s_n\} = \left\{1 + \frac{(-1)^n}{n}\right\}$ seem to be getting closer together.

When we think of $s_n = s(n)$ as a function, we can take its limit as $n \to \infty$, just as we did with functions in Calc I.

If

$$\lim_{n\to\infty} s_n = \text{ a number}$$

then we say $\{s_n\}$ **converges**. Else, we say the sequence **diverges**.

A sequence is **bounded** means there are numbers K, M, such that for all values of n,

$$K \leq s_n \leq M.$$

A sequence is **monotone** means for large enough n (we say $n >> 0$), either $s_{n+1} \leq s_n$ for all n, or $s_{n+1} \geq s_n$ for all n.

In other words, a sequence is monotone means its terms eventually get smaller and smaller or bigger and bigger (or they stay the same).

Theorem (Monotone Convergence Theorem)

A monotone, bounded sequence converges.

Example

The sequence

$$s = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} = \left\{\frac{1}{n}\right\}$$

converges because $0 \leq \frac{1}{n} \leq 1$ for all n (bounded) and $\frac{1}{n+1} \leq \frac{1}{n}$ for all n (monotone).

4 Intro to infinite series

Series

A **series** is an infinite sum of numbers

$$S = a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n.$$

Series don't have to start at $n = 1$, they can start at any integer n (same thing is true for sequences).

The numbers a_n are called the **terms** of the series. The **partial sums** of the series are the numbers

$$S_n = a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

Example (Zeno's Dichotomy Paradox)

$\star$ For a more detailed description of this paradox, see the Stanford Encyclopedia of Philosophy.

Suppose I want to walk across a room. To get from one side to the other, I need to first walk halfway. Then I need to walk half of the remaining distance. Then half of that remaining distance. And so forth.

Given that I must complete an infinite number of tasks, walking half of a distance each time, how is it that I can make it to the other side of the room in finite time?

Soln: The scenario can be modeled using the series

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

Each time I walk half of the remaining distance, I get closer to 1, the full length of the room. Thus we expect S to converge to 1.

A series S **converges** means the sequence of its partial sums converges. That is, we want

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^n} = 1.$$

How do we write $S_n = \frac{1}{2} + \cdots + \frac{1}{n}$ in such a way that we can take the limit? In this example, there is a trick. First, note that

$$\frac{1}{2}S_n = \underbrace{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n}}_{S_n - \frac{1}{2}} + \frac{1}{2^{n+1}}.$$

Therefore,

$$S_n - \frac{1}{2}S_n = S_n - \left(S_n - \frac{1}{2} + \frac{1}{2^{n+1}}\right).$$

Now simplify and solve for S_n:

$$\begin{aligned}
S_n - \frac{1}{2}S_n &= \cancel{S_n} - \left(\cancel{S_n} - \frac{1}{2} + \frac{1}{2^{n+1}}\right)\\
\left(1 - \frac{1}{2}\right) S_n &= \frac{1}{2} - \frac{1}{2^{n+1}}\\
\frac{1}{2}S_n &= \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2^n}\right)\\
\implies S_n &= \frac{\cancel{\frac{1}{2}} - \cancel{\frac{1}{2}}\left(\frac{1}{2^n}\right)}{\cancel{\frac{1}{2}}}\\
&= 1 - \frac{1}{2^n}
\end{aligned}$$

Taking the limit, we get $\lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1$, as expected.

Geometric series

The reason the trick for finding S_n in the previous example worked is that S was an example of a geometric series.

In general, a **geometric series** is a series of the form

$$S = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}.$$

In the previous example, $a = \frac{1}{2}$ and $r = \frac{1}{2}$. We also had

$$a_n = \frac{1}{2^n} = \underbrace{\frac{1}{2}}_{a}\underbrace{\left(\frac{1}{2}\right)^{n-1}}_{r^{n-1}}, \text{ and}$$

$$S_n = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \cdots + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}.$$

Using a similar trick as in the previous example, we can conclude that the partial sums of a a geometric series are

$$S_n = a + ar + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

($\star$ The textbook actually derives this formula.)

The sum of the geometric series is the limit of its sequence of partial sums, so we have

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} \text{ provided } r \neq 1.$$

Q: Why do we need $r \neq 1$? What happens to the series when $r = 1$?

In order for the limit to exist, we need r^n to converge. This is because

$$\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \lim_{n\to\infty} (1-r^n).$$

The only way this can happen is if r is a fraction, that is, $|r| < 1$. Then $r^n \to 0$.

The conclusion is that the geometric series

$$S = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

- converges to $\frac{a}{1-r}$ when $|r| < 1$.
- diverges otherwise.

$\star$ See §9.2 Example 1 parts (b) and (c).

4 Intro to infinite series

Sequence of partial sums

When geometric series converge, we can compute exactly what they converge to. For general series we cannot do this, but we can predict whether a series converges or not, using techniques similar to those in §7.7.

The most direct way to predict the convergence of a series is to compute the limit of its partial sums. If the limit exists, then in that case we know exactly what it converges to.

Example (§9.3 #1)

Does $S = \sum_{n=1}^{\infty} n$ converge or diverge?

Soln: The partial sums are

$$S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(the above formula is explaind ~here~). We have

$$\lim_{n\to\infty} \frac{n(n+1)}{2} = \frac{1}{2}\infty \cdot \infty = \infty,$$

so the series diverges.

The Divergence Test

Theorem (The Divergence Test)

If $\lim_{n\to\infty} a_n \neq 0$ *then the series* $S = \sum_{n=1}^{\infty} a_n$ *diverges.*

- Note that we are taking the limit of the *terms* of the series, and not of the partial sums.
- If the a_ns go to infinity, we are adding larger and larger numbers together and cannot possibly get a finite number.

- On the other hand, if $a_n \to L$, a number not equal to 0, then

$$S = \sum_{n=1}^{\infty} a_n \approx \sum_{n=1}^{\infty} L = L + L + L + \cdots = \infty.$$

The series in the previous example diverges, by the Divergence Test, since $\lim_{n\to\infty} n = \infty$. That is, the terms grow without bound and so the sum of them must be infinite.

The Integral Test

Recall, the terms a_n of a series can be thought of as a function $f(n)$, where the inputs are integers.

If we can make the inputs all real numbers, then we have a function $f(x)$. (Convention says that an x represents the input of a real number, while an n represents the input of an integer.)

Theorem (The Integral Test)

Suppose $a_n = f(n)$, *where* $f(x)$ *is positive and decreasing. Then the series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if the integral* $\int_1^{\infty} f(x)\, dx$ *converges.*

The idea is that we can write a Riemann sum of the integral whose terms are the same as those of the series. Then we can compare the area given by the Riemann sum to the area under the curve $f(x)$.

Example (The Harmonic Series)

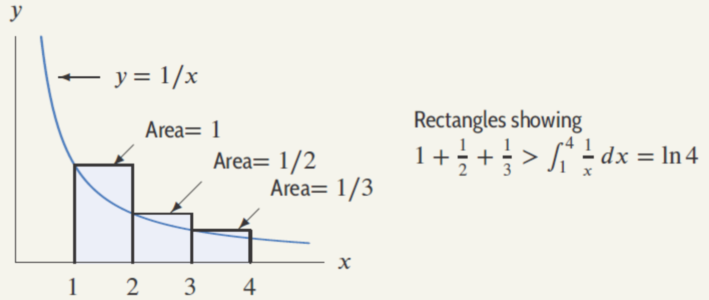
Determine the convergence of $S = \sum_{n=1}^{\infty} \frac{1}{n}$.

Soln: We use the Divergence Test first.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Unfortunately, the Divergence Test does not let us conclude that the harmonic series converges.

On the other hand, we can use the Integral Test since $f(x) = \frac{1}{x}$ is positive and decreasing. Here is why it works:



Graphic: Comparing the harmonic series to $\int_1^\infty \frac{1}{x}\,dx$.

The rectangles have thickness $\Delta x = 1$ so their area is just the function evaluated at the endpoints n. The Riemann sum is an overestimate. We have

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{Riemann sum}} > \int_1^{\infty} \frac{1}{x}\, dx$$

$$= \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\, dx$$

$$= \lim_{b\to\infty} \ln b = \infty.$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\star$ See §9.3 Example 4 for a series that converges by comparing to an integral.

The p-series Test

Theorem (p-series Test)

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

Compare to the useful integrals slide.

The p-series Test can be proven using the Integral Test.
$\star$ See §9.3 Example 5.

4 Intro to infinite series

Application problems

- (1) *§9.1 #63*
- (2)

(1) *§9.1 #63*

The Fibonacci sequence, first studied by the thirteenth-century Italian mathematician Leonardo di Pisa, also known as Fibonacci, is defined recursively by

$$F_n = F_{n-1} + F_{n-2} \text{ for } n > 2 \text{ and } F_1 = F_2 = 1.$$

The Fibonacci sequence occurs in many branches of mathematics and can be found in patterns of plant growth (**phyllotaxis**).

$\star$ See ˜this article˜ for more about Fibonacci in nature and ˜this article˜ for more about the Fibonacci sequence!

(a) Find the first 12 terms.

Soln: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

(b) Show that the sequence of successive ratios $\frac{F_{n+1}}{F_n}$ appears to be a number Φ satisfying the equation $\Phi^2 = \Phi + 1$. (The number Φ was known as the **golden ratio** to the ancient Greeks.)

Soln: We use a calculator to compute the first several terms of the sequence $s = \left\{ \frac{F_{n+1}}{F_n} \right\}$.

$$s_1 = \frac{F_2}{F_1} = \frac{1}{1} = 1$$
$$s_2 = \frac{F_3}{F_2} = \frac{2}{1} = 2$$
$$s_3 = \frac{F_4}{F_3} = \frac{3}{2} = 1.5$$
$$s_4 = \frac{F_5}{F_4} = \frac{5}{3} \approx 1.66666666667$$
$$s_5 = \frac{F_6}{F_5} = \frac{8}{5} = 1.6$$
$$s_6 = \frac{F_7}{F_6} = \frac{13}{8} = 1.625$$

$$s_8 = \frac{F_9}{F_8} = \frac{21}{13} \approx 1.61538461538$$

$$s_9 = \frac{F_{10}}{F_9} = \frac{34}{21} \approx 1.61904761905$$

$$s_{10} = \frac{F_{11}}{F_{10}} = \frac{55}{34} \approx 1.61764705882$$

$$s_{11} = \frac{F_{12}}{F_{11}} = \frac{89}{55} \approx 1.61818181818$$

$$s_{12} = \frac{F_{13}}{F_{12}} = \frac{144}{89} \approx 1.61797752809$$

$$s_{13} = \frac{F_{14}}{F_{13}} = \frac{233}{144} \approx 1.61805555556$$

$$s_{14} = \frac{F_{15}}{F_{14}} = \frac{377}{233} \approx 1.61802575107$$

... The ratios appear to be approaching an number close to 1.618. Now we check:

$$(1.618)^2 = 2.167924$$
$$(1.618) + 1 = 2.168$$

$\star$ Φ is actually equal to $\frac{1+\sqrt{5}}{2} = 1.6180339887\ldots.$

($\star$ Bonus slide!) Check:

$$\begin{aligned}\Phi^2 - \Phi - 1 &= \left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1+\sqrt{5}}{2}\right) - 1 \\ &= \frac{1+2\sqrt{5}+5}{4} - \frac{2(1+\sqrt{5})}{4} - \frac{4}{4} \\ &= \frac{1+\cancel{2\sqrt{5}}+5-2\cancel{-2\sqrt{5}}-4}{4} \\ &= \frac{1+5-2-4}{4} = 0\end{aligned}$$

($\star$ Bonus bonus slide!)

Q: Where does the equation $\Phi^2 - \Phi - 1 = 0$ (or $\Phi^2 = \Phi + 1$) come from?

Ans: Start with the recursion formula to derive the formula for the ratios, then take the limit:

$$F_{n+1} = F_n + F_{n-1}$$

$$\frac{F_{n+1}}{F_n} = \frac{F_n}{F_n} + \frac{F_{n-1}}{F_n}\left(\frac{\frac{1}{F_{n-1}}}{\frac{1}{F_{n-1}}}\right)$$

$$\implies \lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \lim_{n\to\infty}\left(1 + \frac{1}{\frac{F_n}{F_{n-1}}}\right)$$

$$= \Phi = 1 + \frac{1}{\Phi}$$

$$\implies \Phi^2 = \Phi + 1$$

(2)

The Riemann-zeta function ζ is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

and is used in number theory to study the distribution of prime numbers. (You can read a little history about it at ˜this˜ website.)

(a) *(§11.4 #34, Stewart)* Leonhard Euler was able to calculate the exact sum of the p-series with $p = 2$:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Use this fact to find the sum of the series

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$
2. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

Soln:

1. We have

$$\sum_{n=2}^{\infty} \frac{1}{n^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) - \frac{1}{1^2}$$
$$= \frac{\pi^2}{6} - 1.$$

2. Change the indices in the summation. Note that

$$\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots \text{ and}$$
$$\sum_{n=4}^{\infty} \frac{1}{n^2} = \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots .$$

And,

$$\sum_{n=4}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{1} - \frac{1}{2^2} - \frac{1}{3^2} = \frac{\pi^2}{6} - \frac{49}{36}.$$

3.

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}$$

(b) *(§11.4 #35, Stewart)* Euler also found the sum of the p-series with $p = 4$:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use Euler's result to find the sum of the series

1. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$.
2. $\displaystyle\sum_{k=5}^{\infty} \frac{1}{(k-2)^4}$.

Soln:

1.
$$\sum_{n=1}^{\infty}\left(\frac{3}{n}\right)^4 = 3^4\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{81\pi^4}{90} = \frac{9\pi^4}{10}$$

2. Again, we change indices. Let $n = k-2$. Then $n(5) = 5-2 = 3$ and we have

$$\sum_{k=5}^{\infty}\frac{1}{(k-2)^4} = \sum_{n=3}^{\infty}\frac{1}{n^4}$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^4} - \frac{1}{1^4} - \frac{1}{2^4} = \frac{\pi^4}{90} - \frac{17}{16}.$$

Week 5 overview

5 Convergence tests for series

§9.4 Tests for convergence

- The Comparison Test
- The Limit Comparison Test
- The Ratio Test
- The Alternating Series Test
- Guidelines for choosing convergence tests
- More examples

Application problems

- *(§11.6 #50, Stewart)*

5 Convergence tests for series

§9.4 Tests for convergence

- The Comparison Test
- The Limit Comparison Test
- The Ratio Test
- The Alternating Series Test
- Guidelines for choosing convergence tests
- More examples

Application problems

The Comparison Test

Just as we did with improper integrals, we can compare the terms of two series to determine convergence.

Theorem (The Comparison Test)

Given series $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$*, suppose* $0 \leq a_n \leq b_n$ *for all* $n >> 0$*.*

- *If* $\sum_{n=1}^{\infty} b_n$ *converges, then so does* $\sum_{n=1}^{\infty} a_n$*.*
- *If* $\sum_{n=1}^{\infty} a_n$ *diverges, then so does* $\sum_{n=1}^{\infty} b_n$*.*

When considering a series $\sum_{n=1}^{\infty} a_n$, we can look at another series $\sum_{n=1}^{\infty} b_n$ whose convergence we know, and use algebra to compare the terms.

Example (§9.4 #14)

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$ converges.

Soln: When n is really large, the 2^ns nearly cancel each other out and what's left is a factor of n in the denominator (the 1s become negligible for large enough n).

Thus we compare to $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

We need to show $\frac{2^n+1}{n2^n-1} > \frac{1}{n}$ for $n >> 0$.

The 2^ns almost cancel out so we write

$$\frac{1}{n} = \frac{2^n+1}{n(2^n+1)},$$

so that they do. Now we compare the denominator $n(2^n+1)$ to the denominator $n2^n-1$. We have

$$n(2^n+1) = n2^n + n > n2^n - 1.$$

We conclude

$$\frac{2^n+1}{n2^n-1} > \frac{2^n+1}{n(2^n+1)} = \frac{1}{n}.$$

Therefore, $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$ diverges.

The Limit Comparison Test

As seen in the previous example, sometimes the algebra can be cumbersome or require ingenuity. But if we can find series $\sum_{n=1}^{\infty} b_n$ that "behaves" like $\sum_{n=1}^{\infty} a_n$, then there is a shortcut to determining convergence.

Theorem (The Limit Comparison Test)

Suppose $a_n > 0$ *and* $b_n > 0$ *for all* n. *If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0 \textit{ and } c \neq \pm\infty$$

then $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *either both converge or both diverge.*

- In the limit, we can also take $\frac{b_n}{a_n}$ and get the same conclusion.

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$ converges.

Soln: As we saw before, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ "behaves" like our series $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{2^n+1}{n2^n-1}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{n(2^n+1)}{n2^n-1} \\
&= \lim_{n\to\infty} \frac{\cancel{n}(2^n+1)}{\cancel{n}(2^n-\frac{1}{n})} \\
&= \lim_{n\to\infty} \frac{2^n+1}{2^n-\cancelto{0}{\frac{1}{n}}} \\
&= \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{2^n}} + \lim_{n\to\infty} \cancelto{0}{\frac{1}{2^n}} \\
&= 1.
\end{aligned}$$

Since we saw the harmonic series diverges, so must $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$.

Example (§9.4 #28)

Determine the convergence of $\sum_{n=1}^{\infty} \frac{2^n}{3^n - 1}$.

Soln: When $n >> 0$, the 1 in the denominator becomes negligible and the series behaves like the geometric series

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

(where in the geometric series, $a = r = \frac{2}{3}$). The geometric series converges, since $\left|\frac{2}{3}\right| < 1$.

Use the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{\frac{2^n}{3^n-1}}{\frac{2^n}{3^n}} = \lim_{n\to\infty} \frac{3^n \cancel{2^n}}{\cancel{2^n}(3^n-1)}$$

$$= \lim_{n\to\infty} \frac{3^n}{3^n-1}$$

$$= \lim_{n\to\infty} \frac{\cancel{3^n}}{\cancel{3^n}\left(1-\cancelto{0}{\frac{1}{3^n}}\right)}$$

$$= 1 > 0.$$

Therefore, $\sum_{n=1}^{\infty} \frac{2^n}{3^n-1}$ converges.

The Ratio Test

A geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ has the property that for all n,

$$\frac{ar^n}{ar^{n-1}} = \frac{\cancel{a}r^{\cancel{n}}}{\cancel{a}\cancel{r^{n-1}}} = r.$$

If the absolute value of that ratio between successive terms is less than 1, then the series converges.

Some series behave like geometric series when $n >> 0$, that is, if the (absolute value of the) ratio between successive terms approaches a number less than 1, then the series also converges.

Theorem (The Ratio Test)

Suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$.

- *If* $L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *converges.*
- *If* $L > 1$, *then the series diverges.*
- *If* $L = 1$, *then the test fails to give any information about the series.*

$\star$ §9.4 Example 7 gives two examples where $L = 1$; one series diverges while the other converges.

The Ratio Test is useful for series where factorials appear, or where n appears in the exponent.

A ˜**factorial**˜ $n!$ is the product of all integers less than or equal to n:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

Factorials grow *fast*. They also cancel nicely. For example,

$$\frac{10!}{8!} = \frac{10 \cdot 9 \cdot \cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 10 \cdot 9 = 90.$$

Example (§9.4 #18)

Determine whether the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges.

Soln: Use the Ratio Test; make sure in the numerator to plug $n+1$ in for n exactly:

$$\lim_{n\to\infty} \frac{\left|\frac{((n+1)!)^2}{(2(n+1))!}\right|}{\left|\frac{(n!)^2}{(2n)!}\right|} = \lim_{n\to\infty} \frac{(2n)!((n+1)!)^2}{(n!)^2(2(n+1))!}$$

The absolute values go away, since everything is positive.

To begin cancelling, we use the facts that:

$$(n+1)! = (n+1)n! \quad \text{and}$$
$$((2(n+1))! = (2n+2)! = (2n+2)(2n+1)(2n)!$$

Replacing $(n+1)!$ and $(2(n+1))!$, we have

$$\lim_{n\to\infty} \frac{(2n)!((n+1)!)^2}{(n!)^2(2(n+1))!} = \lim_{n\to\infty} \frac{\cancel{(2n)!}((n+1)n!)^2}{(n!)^2(2n+2)(2n+1)\cancel{(2n)!}}$$
$$= \lim_{n\to\infty} \frac{(n+1)^2\cancel{(n!)^2}}{\cancel{(n!)^2}(2n+2)(2n+1)}$$
$$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2}.$$

Now we evaluate the limit as in Calc I. The highest exponent that appears in the denominator is 2, so multiply top and bottom by $\frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty} \frac{1+\cancelto{0}{\frac{2}{n}}+\cancelto{0}{\frac{1}{n^2}}}{4+\cancelto{0}{\frac{6}{n}}+\cancelto{0}{\frac{2}{n^2}}}$$

$$= \frac{1}{4} < 1,$$

so we conclude the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges.

The Alternating Series Test

A series whose terms alternate between positive and negative is called an **alternating series**.

In its summation form, a factor of $(-1)^n$ or $(-1)^{n\pm 1}$ usually appears. Or we might see $\cos n\pi$, since it also alternates between ± 1.

Example

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is an alternating series, since

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} = \frac{1}{2 \ln 2} - \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} - \cdots + \frac{(-1)^n}{n \ln n} + \cdots .$$

(Recall, the indexing doesn't have to start at $n = 1$.)

Theorem (The Alternating Series Test)

An alternating series of the form

$$a_1 - a_2 + a_3 - \cdots (-1)^{n\pm 1} a_n + \cdots \quad \text{or}$$
$$-a_1 + a_2 - a_3 + \cdots (-1)^n a_n + \cdots$$

converges if

- $\lim_{n\to\infty} a_n = 0$ *and*
- $0 < a_{n+1} < a_n$ *for* $n >> 0$.

In words, if the a_ns get small enough fast enough then we are adding, then subtracting, a smaller and smaller amount. It makes sense then that the series converges under these conditions.

Also observe that the first condition in the Alternating Series Test is the same as the one in the Divergence Test. Always check it first, to save time.

If one or both conditions fail then the series may or may not converge.

$\star$ ˜StackExchange has a forum˜ with examples of convergent series for which the Alternating Series Test fails.

A series $\sum_{n=1}^{\infty} a_n$ is

- **conditionally convergent** means it converges but $\sum_{n=1}^{\infty} |a_n|$ does not.
- **absolutely convergent** means both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ converge.

$\star$ The ~Riemann series theorem~ says any conditionally convergent series can converge to any number, depending on how the terms are ordered.

Theorem

If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

Guidelines for choosing convergence tests

(1) Do the Divergence Test first. Even if it fails, it is the quickest to do.

(2) If it's an alternating series use the Alternating Series Test.

(3) Is it a geometric or a p-series? We know the convergence properties of those types of series.

(4) Does it behave like a geometric or p-series? Use the Comparison Test if the algebraic manipulations are straightforward, otherwise use the Limit Comparison Test.

(5) If it has a factorial or n in the exponent, use the Ratio Test.

Never use the Ratio Test with a rational function or an algebraic function (an **algebraic function** is one that looks like a rational function, but where radical signs may appear). Comparison and Limit Comparison are better for those types of series.

(6) If sine or cosine appears, check for absolute convergence. Unless there is a $\cos n\pi$, in which case the series is actually an alternating series.

(7) Check if the Integral Test works.

More examples

Example (§9.4 #72)

Determine the convergence of $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$.

Soln: When $n >> 0$, $a_n \approx \frac{n^2}{n^2} = 1$. We compare to the series $\sum_{n=1}^{\infty} 1$, which diverges. Unfortunately,

$$\frac{n^2}{n^2+1} < \frac{n^2}{n^2} = 1 \text{ for all } n.$$

We want $\frac{n^2}{n^2+1} > 1$ in order to conclude divergence!

To avoid this problem, we use the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^2+1}}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{1+\cancelto{0}{\frac{1}{n^2}}} = 1,$$

therefore $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$ diverges.

On the other hand, the Divergence Test shows right away that the series diverges.

Example (§9.4 #73)

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1+3^n}{4^n}$.

Soln: When $n >> 0$, the 1 in the numerator becomes neglible and the terms approach $\frac{3^n}{4^n}$. We can verify this using the Ratio Test.

$$\lim_{n\to\infty} \frac{\left|\frac{1+3^{n+1}}{\cancel{4^{n+1}}}\right|}{\left|\frac{1+3^n}{\cancel{4^n}}\right|} = \lim_{n\to\infty} \frac{1+3^{n+1}}{4(1+3^n)}\left(\frac{\frac{1}{3^n}}{\frac{1}{3^n}}\right)$$

$$= \lim_{n\to\infty} \frac{\cancelto{0}{\frac{1}{3^n}} + 3}{4\left(\cancelto{0}{\frac{1}{3^n}} + 1\right)} = \frac{3}{4} < 1$$

Therefore, the series converges.

Example (§9.4 #77)

Determine the convergence of $\sum_{n=0}^{\infty} e^{-n}$.

Soln: The Divergence Test fails since

$$\lim_{n\to\infty} e^{-n} = \lim_{n\to\infty} \frac{1}{\cancel{e^n}_{\nearrow 0}} = 0.$$

However, $\sum_{n=0}^{\infty} \frac{1}{e^n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ is a geometric series with $a = 1$ and $r = \frac{1}{e}$. Since $e \approx 2.7$, $\frac{1}{e} < 1$. Using the formula for geometric series, we have that the series converges to

$$\sum_{n=0}^{\infty} e^{-n} = \frac{1}{1 - \frac{1}{e}} \approx 1.582.$$

Example (§9.4 #89)

Determine the convergence of $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

Soln: The terms of the series change between positive and negative, but the series is not alternating. We check for absolute convergence. Since $\sin n$ is between -1 and 1, $|\sin n| \leq 1$. Therefore

$$\frac{|\sin n|}{|n^2|} \leq \frac{1}{n^2}.$$

Thus the absolute series converges, because the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ does. We conclude $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

5 Convergence tests for series

§9.4 Tests for convergence

Application problems

- *(§11.6 #50, Stewart)*

(§11.6 #50, Stewart)

Around 1910, Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}.$$

William Gosper used this series in 1985 to compute the first 17 million digits of π.

(a) Verify that this series is convergent.

(b) How many correct decimal places of π do you get if you use just the first term of the series? What if you use two terms?

Soln:

(a) The constant in front of the summation does not affect convergence, so we ignore it. Since factorials appear in the series, we use the Ratio Test.

$$\lim_{n\to\infty} \frac{\left|\frac{(4(n+1))!(1103+26390(n+1))}{((n+1)!)^4 396^{4(n+1)}}\right|}{\left|\frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}}\right|}$$

$$= \lim_{n\to\infty} \frac{(n!)^4 396^{4n}(4(n+1))!(1103+26390(n+1))}{((n+1)!)^4 396^{4(n+1)}(4n)!(1103+26390n)}$$

A lot of cancelling happens.

- $\dfrac{(n!)^4}{((n+1)!)^4} = \left(\dfrac{n!}{(n+1)!}\right)^4 = \dfrac{1}{(n+1)^4}$
- $\dfrac{396^{4n}}{396^{4(n+1)}} = \dfrac{1}{396^4}$
- $\dfrac{(4(n+1))!}{(4n)!} = (4n+4)(4n+3)(4n+2)(4n+1)$

Simplifying $(1103 + 26390(n+1))$, the limit becomes

$$\lim_{n\to\infty} \frac{(4n+4)(4n+3)(4n+2)(4n+1)(27493+26390n)}{(n+1)^4 396^4 (1103+26390n)}.$$

To find the limit, notice that both the numerator and the denominator are degree 5 in n. So we take the ratio of the coefficients.

The ratio of the coefficients of n^5 is

$$\frac{4^4 \cdot \cancel{26390}}{396^4 \cdot \cancel{26390}} < 1,$$

so by the Ratio Test, the series converges!

(b) Compare the reciprocal of the first term of the series to π:

$$\frac{9801}{2\sqrt{2}}\frac{(0!)^4 394^{4(0)}}{(4(0))!(1103+26390(0))} \approx 3.14159273001$$

$$\pi \approx 3.14159265359$$

The reciprocal of the second partial sum (one can use ˜Desmos˜ to compute) is:

$$\frac{1}{\frac{2\sqrt{2}}{9801}\sum_{n=0}^{1}\frac{(4n)!(1102+26390n)}{(n!)^4 396^{4n}}} \approx 3.14159265359$$

Week 6 overview

6 Power series & intro to Taylor series

§9.5 Power series and interval of convergence

- Power series
- Radius of convergence
- Interval of convergence
- Indexing even and odd terms
- **Announcements**

§10.1 Taylor polynomials

- Recall: linear approximations
- Tangent parabolas and higher degree curves
- Accuracy of Taylor polynomials

Application problems

- *(§10.1 #48)*

6 Power series & intro to Taylor series

§9.5 Power series and interval of convergence

- Power series
- Radius of convergence
- Interval of convergence
- Indexing even and odd terms
- **Announcements**

§10.1 Taylor polynomials

Application problems

Power series

A **power series** about a is a series of the form

$$C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} C_n(x-a)^n,$$

where the C_ns are constants and a is a fixed number.

Example (§9.5 #1-4)

Which ones are power series?

(1) $x - x^3 + x^6 - x^{10} + x^{15} - \cdots$

(2) $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \cdots$

(3) $1 + x + (x-1)^2 + (x-2)^3 + (x-3)^4 + \cdots$

(4) $x^7 + x + 2$

Soln:

(1) Is a power series about 0, where $C_0 = 0$, $C_2 = 0$, $C_4 = C_5 = 0$, etc.

(2) Not a power series because x appears in the denominator.

(3) Not a power series because it is impossible to determine a.

(4) Is a power series about 0 with $C_0 = 2$, $C_1 = 1$, $C_7 = 1$, and all other C_ns equal to 0.

Radius of convergence

Power series always converge when $x = a$; they converge to C_0.

Power series don't always converge for all x.

- If a power series does converge for all x, we say its radius of convergence is ∞.
- If it converges at $x = a$ and diverges at all other values of x, we say its radius of convergence is 0.

To find the values of x where a power series converges, we use the Ratio Test.

$$\begin{aligned}\lim_{n\to\infty} \frac{|C_{n+1}(x-a)^{n+1}|}{|C_n(x-a)^n|} &= \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}|x-a| \\ &= |x-a| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}.\end{aligned}$$

For convergence, we want this limit to be less than 1, as per the Ratio Test. We consider the factor $\lim_{n\to\infty} \frac{C_{n+1}}{C_n}$.

- If $\lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}$ does not exist, then there is no possible way multiplying by $|x-a|$ will give a number less than 1, for any value of $x \neq a$.

  In this case the radius of convergence is 0.
- If $\lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = 0$, then for any value of x, $|x-a| \cdot 0 = 0 < 1$, and so the power series converges.

  Thus the radius of convergence is ∞.

- If $\lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}$ exists and equals a positive number K, then we want

$$|x-a| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = |x-a|K < 1$$
$$\implies |x-a| < \frac{1}{K}.$$

The number $R = \frac{1}{K}$ is the **radius of convergence** for the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$.

All values of x within a distance R of a give convergence for the power series; values of x greater than that distance give divergence.

Example (§9.5 #12)

Find the radius of convergence of $\sum_{n=0}^{\infty}(5x)^n$.

Soln: The power series is

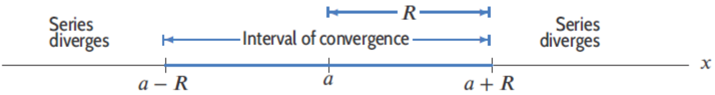
$$\sum_{n=0}^{\infty}(5x)^n = 1 + 5x + 25x^2 + 125x^3 + 625x^4 + \cdots\ ,$$

so $C_n = 5^n$.

$$\lim_{n\to\infty}\frac{|C_{n+1}|}{|C_n|} = \lim_{n\to\infty}\frac{5^{n+1}}{5^n} = 5 \leftarrow K.$$

So the radius of convergence is $\frac{1}{K} = \frac{1}{5}$.

Interval of convergence



Graphic: Radius of convergence, R, determines an interval centered at $x = a$, where the series converges.

Recall, the Ratio Test is inconclusive when the limit equals 1. In the context of power series, this means we have

$$|x-a|K = 1 \implies |x-a| = \frac{1}{K} = R.$$

In other words, we cannot determine convergence when $x = a \pm R$.

The **interval of convergence** for a power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ is the interval between $a-R$ and $a+R$, including any endpoint where the series converges.

To determine the interval of convergence we first find the radius of convergence, then check the endpoints by hand, using techniques from §9.3-9.4.

Example (§9.5 #27)

Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$.

Soln: The power series is centered about $a = 0$ and the coefficients are $C_n = \frac{1}{3^n}$. First find the radius of convergence.

$$\lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = \lim_{n\to\infty} \frac{\frac{1}{3^{n+1}}}{\frac{1}{3^n}}$$

$$= \lim_{n\to\infty} \frac{3^n}{3^{n+1}} = \frac{1}{3}$$

The radius of convergence is $\frac{1}{\frac{1}{3}} = 3$.

The interval of convergence contains the interval $(0-3, 0+3) = (-3, 3)$ and may or may not include the endpoints ± 3.

We must check convergence for each individual endpoint by plugging in $x = \pm 3$.

$\underline{x = -3}$:

Does the series $\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n}$ converge? We have

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n.$$

By the Divergence Test, this series diverges, and so $x = -3$ is not included in the interval of convergence.

$\underline{x = 3}$:

Does the series $\sum_{n=0}^{\infty} \frac{3^n}{3^n}$ converge? No, because the series simplifies to $\sum_{n=0}^{\infty} 1$, which again diverges by the Divergence Test. So $x = 3$ is not included in the inverval of convergence.

We conclude that the interval of convergence for $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ is $(-3, 3)$.

Indexing even and odd terms

Example (§9.5 #23)

Find the radius of convergence for the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.

Soln: To write the series in closed form (and thus find the general term), use $2n+1$ to skip the even powers of x. Use a $(-1)^n$ to get the alternating signs.

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

⋆ Check this by plugging in $n = 0, 1, \ldots$.

Now use the Ratio Test:

$$\lim_{n\to\infty} \frac{\left|\frac{\cancel{(-1)^{(n+1)}}x^{2(n+1)+1}}{2(n+1)+1}\right|}{\left|\frac{\cancel{(-1)^{n}}x^{2n+1}}{2n+1}\right|} = \lim_{n\to\infty}\left|\frac{(2n+1)x^{2(n+1)+1}}{(2(n+1)+1)x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+1)x^{2n+3}}{(2n+3)x^{2n+1}}\right|$$

$$= |x^2| \lim_{n\to\infty}\left|\frac{2n+1}{2n+3}\right|$$

$$= |x^2| < 1 \implies |x| < 1.$$

Therefore, the radius of convergence is 1.

Announcements

Wednesday 24 February, 2021

- No Week 7 homework (ignore what the Course calendar says).
- **Due:** Week 5 homework (1159p EST), Week 6 homework (Wed 3 Mar)

Recall: linear approximations

From Calc I, we saw that the tangent line to a smooth curve $f(x)$ at a point $x = a$ has the equation

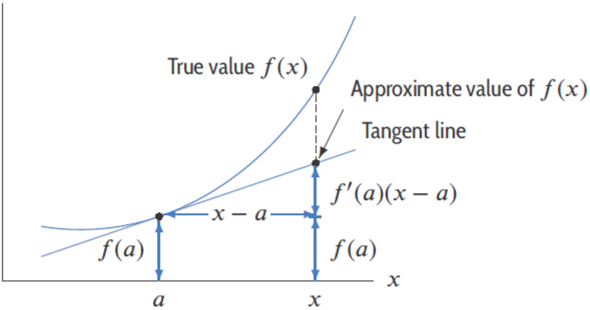
$$y - f(a) = f'(a)(x - a),$$

since it has slope $f'(a)$ and contains the point $(a, f(a))$. We can rewrite this as

$$y = f(a) + f'(a)(x - a).$$

For x-values "near" a, the curve $f(x)$ is pretty close to the tangent line. Thus we have the **linear approximation**

$$f(x) \approx f(a) + f'(a)(x - a).$$



Graphic: Tangent line approximation to $f(x)$ for x near a.

Unfortunately, as we can see from the picture on the previous slide, the tangent line isn't a good enough approximation when x gets far away from a.

The idea behind Taylor polynomials is to "bend" the tangent line so that it fits along the contours of $f(x)$. This makes an approximation that works for x-values farther away from a.

Tangent parabolas and higher degree curves

Here are the criteria we want for a tangent parabola to a smooth curve $f(x)$ at $x = a$.

(1) The parabola should pass through the point $(a, f(a))$.

(2) The slope should be the same as $f'(a)$ at $x = a$.

(3) The concavity of the parabola at $x = a$ should match the concavity of $f(x)$.

Parabolas (centered at the origin) have the form $y = C_0 + C_1 x + C_2 x^2$ for constants C_0, C_1, C_2.

To meet criterion (1), the y-intercept should be $C_0 = f(a)$ and we shift the vertex of the parabola to $x = a$ by replacing x with $x - a$.

The tangent parabola should have the form

$$y = f(a) + C_1(x-a) + C_2(x-a)^2.$$

Now we use criteria (2) and (3) to find C_1 and C_2.

To meet criterion (2), take the derivative of the parabola, plug in $x = a$, and set the result equal to $f'(a)$.

$$\begin{aligned} \frac{dy}{dx} &= C_1 + 2C_2(x-a) \\ \implies \left.\frac{dy}{dx}\right|_{x=a} &= C_1 = f'(a) \end{aligned}$$

This means the tangent parabola has the form

$$y = f(a) + f'(a)(x-a) + C_2(x-a)^2.$$

It makes sense that the equation contains the linear approximation.

Now we apply criterion (3) to find C_2. The concavities of $f(x)$ and the parabola must match at $x = a$, so we want the second derivative of the parabola to be $f''(a)$ at $x = a$.

We take the second derivative, plug in $x = a$, and set the result equal to $f''(a)$ in order to solve for C_2.

$$\frac{d^2y}{dx^2} = 2C_2 \implies \left.\frac{d^2y}{dx^2}\right|_{x=a} = 2C_2 = f''(a)$$

$\implies C_2 = \frac{f''(a)}{2}$.

The tangent parabola is

$$y = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Q: A tangent cubic should give a better approximtion of $f(x)$, for even more values of x farther away from a. What do you think the equation for a tangent cubic will look like?

Ans: We can derive the equation by applying the same criteria for the tangent parabola, plus the requirement that the third derivatives at $x = a$ are equal.

In general, the **Taylor polynomial** of degree n approximating $f(x)$ for x near a is given by

$$\begin{aligned} P_n(x) = f(a) &+ f'(a)(x-a) \\ &+ \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3\cdot 2}(x-a)^3 \\ &+ \frac{f''''(a)}{4\cdot 3\cdot 2}(x-a)^4 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

The linear approximation is $P_1(x)$, the tangent parabola is $P_2(x)$, and the tangent cubic is $P_3(x)$.

Terms of a Taylor polynomial shrink fast! Also, the Taylor approximations get really good, really fast. It's what your calculators and computers use to compute functions like e^x, $\cos x$, and $\sin x$.

$\star$ Try it! Graph the first several Taylor polynomials of $f(x) = \sin x$ for x near 0.

Accuracy of Taylor polynomials

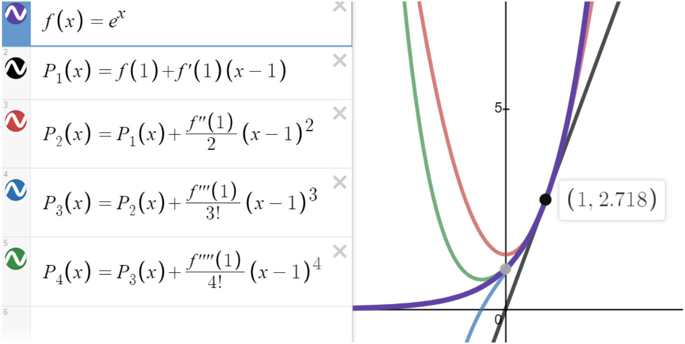
Example (§10.1 #12)

Find the Taylor polynomial of degree $n = 4$ for $f(x) = e^x$ near $a = 1$.

Soln: We have a formula, so we compute the derivatives at 1. All derivatives of e^x are equal to e^x and $e^1 = e$. Therefore the Taylor polynomial is

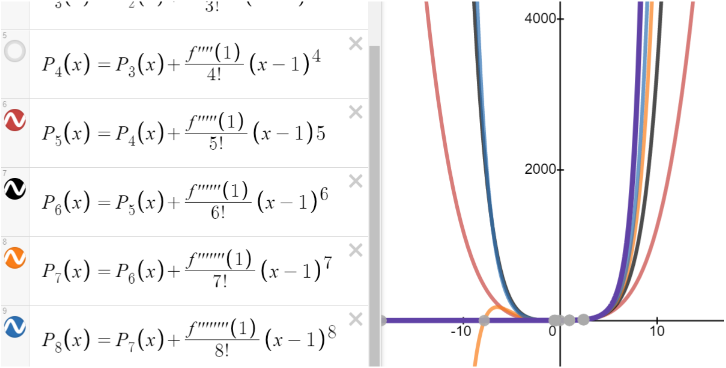
$$P_4(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \frac{e}{4!}(x-1)^4.$$

$P_4(x)$ is the tangent quartic to e^x at $x = 1$. We can visualize this, along with the lower degree approximations, using ~Desmos~.



Graphic: $P_4(x)$ and lower degree Taylor approximations to $f(x) = e^x$ for x near 1.

Using the template from the previous slide, we can investigate he higher degree Taylor approximations to $f(x) = e^x$ near $x = 1$.



Graphic: Higher degree Taylor aprpoximations for e^x.

Notice how as n gets bigger, the Taylor polynomial $P_n(x)$ gets closer to the curve $f(x) = e^x$ for x-values farther and farther away from 1. This example would lead one to believe that

$$\lim_{n\to\infty} P_n(x) = f(x).$$

In other words, if we make an infinite Taylor polynomial for $f(x)$, i.e., a **Taylor series**, then it will exactly equal $f(x)$.

Unfortunately, it's too good to be true!

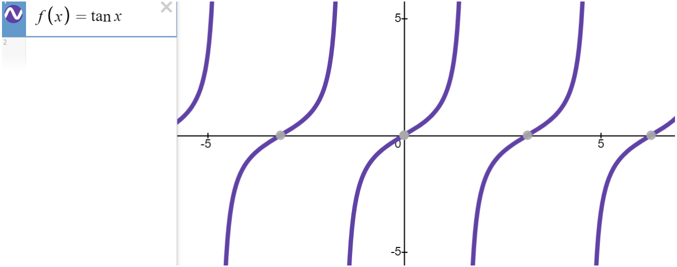
$\star$ However, in §10.2 we will use the fact that Taylor series are power series to find their intervals of convergence.

Example (§10.1 #8)

Find the degree 3 and 4 Taylor polynomials for $f(x) = \tan x$, about the point $x = 0$.

We will again use ~Desmos~ to look at the higher degree Taylor polynomials.

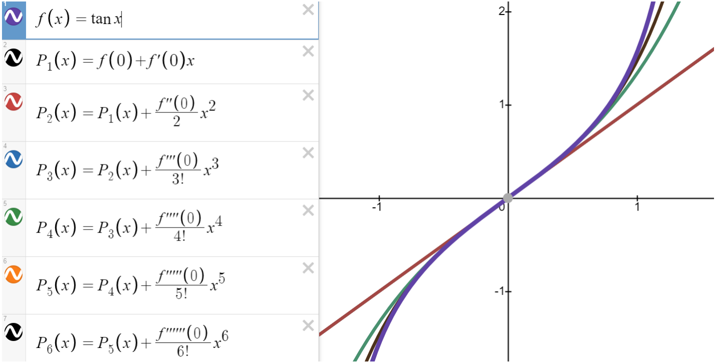
When trying to determine whether $\tan x$ equals its Taylor series we come across a red flag. Recall, the graph of $\tan x$:



Graphic: Graph of $\tan x$.

There is an asymptote at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, etc. But polynomials don't have asymptotes! How can any polynomial perfectly approximate $\tan x$?

In fact, here are the first 6 Taylor polynomials for $\tan x$ about $x = 0$:



Graphic: First 6 Taylor polynomials for $\tan x$ *for* x *near* 0.

From the picture, it looks like the Taylor series will only converge to the piece of $\tan x$ that contains $x = 0$.

6 Power series & intro to Taylor series

§9.5 Power series and interval of convergence

§10.1 Taylor polynomials

Application problems

- *(§10.1 #48)*

(§10.1 #48)

When we model the motion of a pendulum, we sometimes replace the differential equation (a **differential equation** is an equation where derivatives and higher-order derivatives of a function appear alongside the independent variable of the function):

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \quad \text{by} \quad \frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta,$$

where θ is the angle between the pendulum and the vertical (g is the gravitational constant and l is the length of the pendulum).

Explain why, and under what circumstances, it is reasonable to make this replacement.

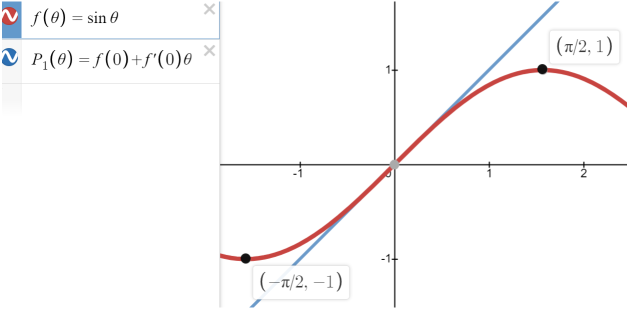
Soln: The problem is claiming that under certain circumstances, $\sin\theta \approx \theta$.

We can verify this by using Taylor approximations. Let $f(\theta) = \sin\theta$. The linear approximation to $f(\theta)$ for θ near 0 is

$$\begin{aligned} f(\theta) &\approx f(0) + f'(0)(\theta - 0) \\ &= \sin(0) + \cos(0)\theta \\ &= \theta. \end{aligned}$$

Now the question is, under what circumstances is this approximation reasonable? In other words, for what values of θ does $\sin\theta \approx \theta$ give a good approximation for θ near 0?

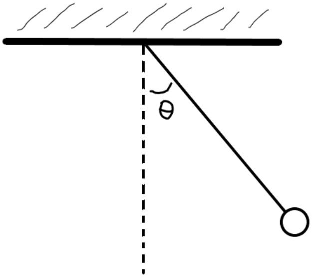
Here is a graph of $f(\theta)$ along with its linear approximation $P_1(\theta)$.



Graphic: $\sin\theta$ *and its linear approximation for* θ *near* 0.

The approximation is pretty good, up until around $\theta = \pm\frac{\pi}{4}$.

What does this mean for the pendulum?



Graphic: Pendulum with angle θ from the vertical.

It means that as long as the pendulum doesn't swing more than 45° from the vertical, the approximation is reasonable. If it does swing higher than that, then a better approximation is needed (e.g., a higher degree Taylor polynomial).

Week 7 overview

7 Taylor series

§10.2 Taylor series

- Maclaurin series
- Binomial series
- Convergence of Taylor series

§10.3 Finding and using Taylor series

- Substitution
- Integration and differentiaion
- Multiplying Taylor series
- More examples

Application problems

- (1) *(§10.2 #59)*
- (2) *(§10.3 #47)*

7 Taylor series

§10.2 Taylor series

- Maclaurin series
- Binomial series
- Convergence of Taylor series

§10.3 Finding and using Taylor series

Application problems

Maclaurin series

The **Taylor series expansion** of a function $f(x)$ about the point $x = a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

$\star$ Note that we can write $f(a) = \frac{f^{(0)}(a)}{0!}(x-a)^0$, $f'(a)(x-a) = \frac{f'(a)}{1!}(x-a)^1$, and that $2 = 2!$.

The **Maclaurin series** of $f(x)$ is its Taylor series expansion about $x = 0$.

Here are some well-known Maclaurin series you should memorize:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

$\star$ Since Taylor series are power series, use techniques from §9.5 to find the intervals of convergence for these series.

Binomial series

A useful Taylor expansion in practice is the Maclaurin series for $f(x) = (1+x)^p$, for *any* number p.

It is particularly useful when p is a fraction or a negative number (note that when p is an integer, $f(x)$ is just a polynomial, and equal to its own Taylor series).

The **binomial series** is given by

$$
\begin{aligned}
(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 \\
+ \frac{p(p-1)(p-2)}{3!}x^3 + \frac{p(p-1)(p-2)(p-3)}{4!}x^4 \\
+ \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}x^n + \cdots
\end{aligned}
$$

$\star$ See §10.2 for a derivation of this formula. See Examples 1-2 for applications of it.

Convergence of Taylor series

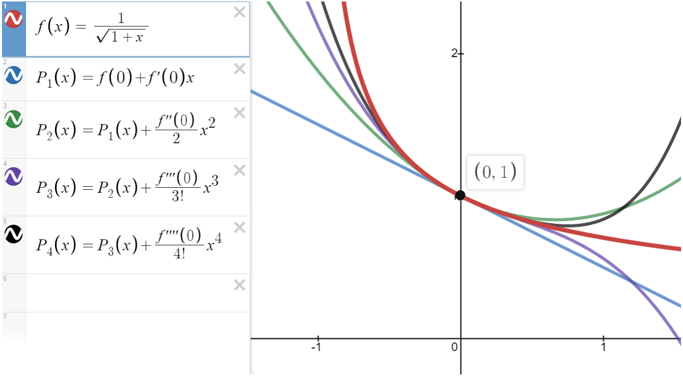
While certain functions equal their Taylor series exactly for all x, most functions you will see in real life converge to $f(x)$ only for an interval.

$\star$ See §10.2 Example 3.

Example (§10.2 #37)

Find the radius of convergence for the Maclaurin series of $f(x) = \frac{1}{\sqrt{1+x}}$.

We first graph the first several Taylor polynomials (about $x=0$) for $f(x)$ to make a guess about its interval of convergence.



Graphic: Taylor polynomials for $f(x) = \frac{1}{\sqrt{1+x}}$.

For $|x| > 1$, the Taylor polynomials seem to diverge from $f(x)$.

The Maclaurin series for $f(x)$ is a binomial series with $p = -\frac{1}{2}$, call it S.

$$S = 1 - \frac{1}{2}x + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}x^2 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3$$
$$+ \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!}x^4 + \cdots + \frac{(-1)^n(2n)!}{(2^n n!)^2}x^n + \cdots$$

$\star$ ˜PhysicsForums˜ has a thread where the formula for the general term for the binomial series of $\frac{1}{\sqrt{1-x}}$ is derived. Replacing x with $-x$ in the formula creates the factor $(-1)^n$. In §10.3 we will use this trick and more!

Use the Ratio Test, as in §9.5:

$$\lim_{n\to\infty} \frac{\left| \frac{\cancel{(-1)^{n+1}}(2(n+1))!}{(2^{n+1}(n+1)!)^2}\, x^{\cancel{n+1}} \right|}{\left| \frac{\cancel{(-1)^{n}}(2n)!}{(2^{n}n!)^2}\, \cancel{x^{n}} \right|} = |x| \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(2(n+1))^2}$$

$$= |x| \lim_{n\to\infty} \frac{2n+1}{2n+2}$$

$$= |x|$$

$\Longrightarrow$ $|x| < 1$ and the radius of convergence is 1.

7 Taylor series

§10.2 Taylor series

§10.3 Finding and using Taylor series

- Substitution
- Integration and differentiaion
- Multiplying Taylor series
- More examples

Application problems

Substitution

In §10.2, we saw the Maclaurin series for $\sin x$, $\cos x$, e^x, and the binomial series. Here are a few other series you will find useful.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \text{ for } -1 < x < 1$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \cdots - \frac{x^n}{n} - \cdots \text{ for } -1 \leq x < 1$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$
$$\text{for } -1 \leq x \leq 1$$

- The first series is called the **geometric series**. (Why?)
- Note that these series do not equal their respective functions unless x is in the given interval of convergence.

With these seven known series, we can derive other Maclaurin series by plugging in expressions for x.

Example (§10.3 #4)

Find the first four non-zero terms of the Maclaurin series for $f(y) = \ln(1 - 2y)$.

Substitute $2y$ for x into the Maclaurin series for $\ln(1-x)$:

$$\ln(1-2y) = -2y - \frac{(2y)^2}{2} - \frac{(2y)^3}{3} - \frac{(2y)^4}{4} + \cdots$$

To find the interval of convergence, we have $-1 \le 2y < 1 \implies -\frac{1}{2} \le y < \frac{1}{2}$.

Integration and differentiation

Recall, the operation of integration is itself a(n infinite) sum. This means we can integrate Taylor series term-by-term to get new Taylor series.

$\star$ See §10.3 Example 3.

What's less obvious though, is that we can differentiate a Taylor series to get a new Taylor series. The following theorem tells us we can.

Theorem

If a Taylor series for $f(x)$ about $x = a$ converges to $f(x)$ for $|x - a| < R$, then the series found by term-by-term differntiation is the Taylor series for $f'(x)$, and converges to $f'(x)$ for all x in the same interval.

This means that when we differentiate a Taylor series, we need to check the endpoints of the interval of convergence.

Example (§10.3 Example 2)

By differentiating term-by-term, one can show that the Maclaurin series expansion of $f(x) = \frac{1}{(1-x)^2}$ is

$$f(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

The theorem says the series converges to $f(x)$ for $-1 < x < 1$, because the series for $\frac{1}{1-x}$ converges on that interval. We check the endpoints:

$\underline{x = -1}$:
The series becomes $1 - 2 + 3 - 4 + 5 - 6 + \cdots$, which diverges.

$\underline{x = 1}$:
The series becomes $1 + 2 + 3 + 4 + 5 + 6 + \cdots$, which diverges.

Therefore the Maclaurin series for $\frac{1}{(1-x)^2}$ converges for $-1 < x < 1$.

Multiplying Taylor series

It is fairly straightforward to find the Taylor series of a function multiplied by a power of x, since we can just multiply term-by-term.

What is less straightforward, however, is mutliplying two Taylor series, i.e. infinite sums together. We use the technique of *gathering terms*.

Example (§10.3 #25)

Find the Maclaurin series for $e^t \cos t$.

The Maclaurin series for e^t and $\cos t$ are known. We just have to multiply them together.

$$\left(1+t+\frac{t^2}{2}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)\left(1-\frac{t^2}{2}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots\right)$$

We gather terms according to the powers of t:

$$\begin{aligned}
1 &: \quad 1\cdot 1\\
t &: \quad t\cdot 1\\
t^2 &: \quad 1\cdot\left(-\frac{t^2}{2}\right)+\frac{t^2}{2}\cdot 1\\
t^3 &: \quad t\cdot\left(-\frac{t^2}{2}\right)+\frac{t^3}{3!}\cdot 1\\
t^4 &: \quad 1\cdot\frac{t^4}{4!}+\frac{t^2}{2}\cdot\left(-\frac{t^2}{2}\right)+\frac{t^4}{4!}\cdot 1\\
&\vdots
\end{aligned}$$

We have

$$\begin{aligned} e^t \cos t &= 1 + t + \left(-\frac{1}{2} + \frac{1}{2}\right) t^2 + \left(-\frac{1}{2} + \frac{1}{3!}\right) t^3 \\ &\quad + \left(\frac{1}{4!} - \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4!}\right) t^4 + \cdots \\ &= 1 + t - \frac{1}{3} t^3 - \frac{1}{6} t^4 + \cdots \end{aligned}$$

⋆ Graph this function along with its fourth degree Taylor approximation to verify this answer makes sense!

More examples

Example (§10.3 #20)

Expand the quantity $\frac{1}{a-r}$ about 0 in terms of the variable $\frac{r}{a}$.

Soln: First use algebra to rewrite the quantity in terms of $\frac{r}{a}$.

$$\frac{1}{a-r}\left(\frac{\frac{1}{a}}{\frac{1}{a}}\right) = \frac{\frac{1}{a}}{1-\frac{r}{a}}$$

$$= \frac{1}{a} \cdot \underbrace{\frac{1}{1-\frac{r}{a}}}_{\text{geometric series}}$$

The geometric series is given by

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots,$$

for $|x| < 1$. We replace x with $\frac{r}{a}$ and multiply by $\frac{1}{a}$:

$$\begin{aligned}\frac{1}{a-r} &= \frac{1}{a} \cdot \frac{1}{1-\frac{r}{a}} \\ &= \frac{1}{a}\left(1 + \frac{r}{a} + \left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^3 + \left(\frac{r}{a}\right)^4 + \cdots\right)\end{aligned}$$

Example (§10.3 #36)

For values of y near 0, put the following functions in increasing order, using their Taylor expansions.

(a) $\ln\left(1+y^2\right)$

(b) $\sin\left(y^2\right)$

(c) $1-\cos y$

Soln: First write the Taylor series expansion for each function.

(a)

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\implies \ln(1+x) = \ln(1-(-x))$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\implies \ln(1+y^2) = (y^2) - \frac{(y^2)^2}{2} + \frac{(y^2)^3}{3} - \frac{(y^2)^4}{4} + \cdots$$

$$= y^2 - \frac{y^4}{2} + \frac{y^6}{3} - \frac{y^8}{4} + \cdots$$

(b)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\implies \sin(y^2) = (y^2) - \frac{(y^2)^3}{3!} + \frac{(y^2)^5}{5!} - \frac{(y^2)^7}{7!} + \cdots$$

$$= y^2 - \frac{y^6}{3!} + \frac{y^{10}}{5!} - \frac{y^{14}}{7!} + \cdots$$

(c)

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

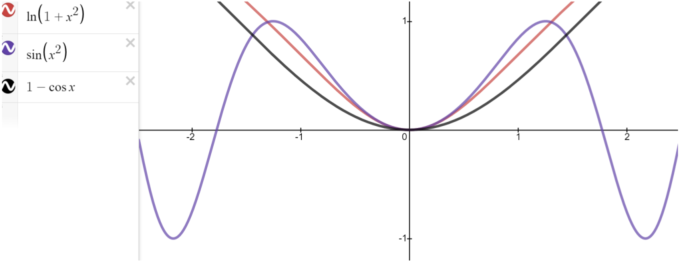
$$\implies 1 - \cos y = 1 - \left(1 - \frac{y^2}{2} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right)$$

$$= \frac{y^2}{2} - \frac{y^4}{4!} + \frac{y^6}{6!} - \cdots$$

To assess which function is bigger near $y = 0$, we can truncate the higher degree terms of the Taylor series expansions and just look at those of degree 4 or less. For y close to 0, we have

$$\underbrace{\frac{y^2}{2} - \frac{y^4}{4!}}_{\approx 1 - \cos y} < \underbrace{y^2 - \frac{y^4}{2}}_{\approx \ln(1+y^2)} < \underbrace{y^2}_{\approx \sin(y^2)} .$$

Finally, we can verify on ~Desmos~ by graphing each function and zooming in near 0.



Graphic: Functions $\ln(1+x^2)$, $\sin(x^2)$, *and* $1-\cos x$, *zoomed in near* $x=0$.

7 Taylor series

§10.2 Taylor series

§10.3 Finding and using Taylor series

Application problems

- (1) *(§10.2 #59)*
- (2) *(§10.3 #47)*

(1) *(§10.2 #59)*

Let $i = \sqrt{-1}$. We define $e^{i\theta}$ by substituting $i\theta$ in the Taylor series for e^x. Use this definition to explain Euler's forumla

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Soln: The Taylor (Maclaurin) series for $e^{i\theta}$ is

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots . \end{aligned}$$

Group the real and imaginary parts of $e^{i\theta}$:

$$
\begin{aligned}
&= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} + \cdots\right) \\
&= \underbrace{\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)}_{\sin\theta} \\
&= \cos\theta + i\sin\theta
\end{aligned}
$$

(2) *(§10.3 #47)*

A hydrogen atom consists of an electron, of mass m, orbiting a proton, of mass M, where m is much smaller than M. The *reduced mass* μ, of the hydrogen atom is defined by

$$\mu = \frac{mM}{m+M}.$$

(a) Show that $\mu \approx m$.

(b) To get a more accurate approximation for μ, express μ as m times a series in $\frac{m}{M}$.

(c) The approximation $\mu \approx m$ is obtained by disregarding all but the constant term in the series. The first-order correction is obtained by including the linear term but no higher terms. If $m \approx \frac{M}{1836}$, by what percentage does including the linear term change the estimate $\mu \approx m$?

Soln:

(a) Since we are assuming $m << M$, the denominator of μ is close to M. Therefore we have

$$\mu = \frac{mM}{m+M} \approx \frac{mM}{M} = m.$$

(b) We first need to rewrite μ to get an expression in $\frac{m}{M}$:

$$\mu = \frac{mM}{m+M}\left(\frac{\frac{1}{M}}{\frac{1}{M}}\right) = \frac{m}{\frac{m}{M}+1}$$

The new expression for μ is a binomial series in $-\frac{m}{M}$. The binomial series is appropriate to use, since $m << M$ means $\frac{m}{M} < 1$.

$$\frac{m}{\frac{m}{M}+1} = m\left(\frac{1}{1-\left(-\frac{m}{M}\right)}\right)$$

The series is:

$$m\left(1+\left(-\frac{m}{M}\right)+\left(-\frac{m}{M}\right)^2+\left(-\frac{m}{M}\right)^3+\left(-\frac{m}{M}\right)^4+\cdots\right)$$

(c) From the previous slide, the constant term of the series is indeed m, so $\mu \approx m$. The linear term is $-\frac{m^2}{M}$ so we compare the linear term to the constant term, making the substitution $m \approx \frac{M}{1836}$:

$$\begin{aligned}\frac{-\frac{m^2}{M}}{m} = -\frac{m}{M} &\approx -\frac{\frac{M}{1836}}{M} \\ &= -\frac{1}{1836} \\ &\approx -0.0005447\end{aligned}$$

So the answer is $\approx -0.05447\%$. Higher terms will only contribute smaller differences so this calculation justifies the use of the approximation $\mu \approx m$.

Week 8 overview

8 Review

Review: Integration

- Week 1: Intro to integration techniques
- Week 2: Advanced integration techniques
- Week 3: Applications of integration

8 Review

Review: Integration

- Week 1: Intro to integration techniques
- Week 2: Advanced integration techniques
- Week 3: Applications of integration

Week 1: Intro to integration techniques

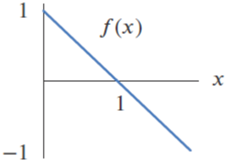
§6.1-6.2, 7.1-7.2

- *(§6.1-6.2)* Know the difference between a definite and an indefinite integral. Know how to graph an antiderivative of a function whose graph is given. Use the Fundamental Theorem of Calculus.
  - ⋆ Questions like: §6.1 #5-13 (odds), 23, 29-33 (odds)

- *(§7.1)* Integration by substitution (including abstract problems, changing bounds on definite integrals).
  - ⋆ Questions like: §7.1 #3-65 (odds), 73-79 (odds), 89-93 (odds), 129
- *(§7.2)* Integration by parts (including abstract problems, using IBP more than once). "Ultraviolet voodoo". LogPoET.
  - ⋆ Questions like: §7.2 #3-31 (odds), 35, 37, 71

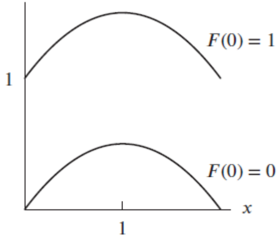
Example (§6.1 #10)

Sketch two functions F such that $F' = f$ (f is given below). In one case let $F(0) = 0$, and in the other case let $F(0) = 1$.



Graphic: Graph of the function f.

Soln:



Graphic: Two functions F, with $F(0) = 0$ and $F(0) = 1$.

Week 2: Advanced integration techniques

§7.3-7.4

- *(§7.3)* Use algebra to prepare an integral for use with a table (completing the square, long division).
  - ⋆ Questions like: §7.3 #15-49 (odds), 53
- *(§7.4)* Integrate using partial fractions.
  - ⋆ Questions like: §7.4 #39-53 (odds)
- *(§7.4, cont.)* Trig substitutions, invoking the triangle.
  - ⋆ Questions like: §7.4 #55-65 (odds)

Example (§7.4 #42)

Find $\int \frac{dz}{z^2+z}\, dz$.

Soln: First factor $z^2 + z = z(z+1)$. Then write

$$\frac{1}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}.$$

To find A and B, get a common denominator on the right hand side. Then match coefficients to get a system of 2 equations in 2 unknowns.

$$\frac{1}{z(z+1)} = \frac{A}{z}\left(\frac{z+1}{z+1}\right) + \frac{B}{z+1}\left(\frac{z}{z}\right)$$
$$\implies 1 = Az + A + Bz$$

We have

$$0 = A + B \implies A = -B$$
$$1 = A \implies B = -1$$

The integral becomes

$$\int \frac{dz}{z^2+z} = \int \frac{1}{z}\,dz + \int \frac{-1}{z+1}\,dz$$
$$= \ln z - \ln(z+1) + C.$$

Example (§7.4 #56)

Find $\int \frac{\sqrt{1-4x^2}}{x^2}\,dx$.

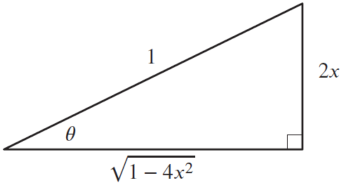
Soln: First rewrite $1-4x^2 = 4\left(\frac{1}{4}-x^2\right)$. Then put $x=\frac{1}{2}\sin\theta \implies dx = \frac{1}{2}\cos\theta$. We have

$$\int \frac{2\sqrt{\frac{1}{4}-x^2}}{x^2}\,dx = 2\int \frac{\sqrt{\frac{1}{4}-\frac{1}{4}\sin^2\theta}}{\frac{1}{4}\sin^2\theta}\cdot\frac{1}{2}\cos\theta\,d\theta$$
$$= 2\int \frac{\cancel{\frac{1}{4}}\cos^2\theta}{\cancel{\frac{1}{4}}\sin^2\theta}\,d\theta$$

Now use the guidance from the table in the text.

$$\begin{aligned}2\int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta &= 2\int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta \\ &= 2\left(\int \csc^2\theta\,d\theta - \int d\theta\right) \\ &= 2(-\cot\theta - \theta) + C\end{aligned}$$

Invoke the triangle:



Graphic: Triangle with $\sin\theta = 2x$.

$$\implies \int \frac{\sqrt{1-4x^2}}{x^2}\,dx$$

$$= 2\left(-\frac{\sqrt{1-4x^2}}{2x} - \arcsin(2x)\right) + C$$

Week 3: Applications of integration

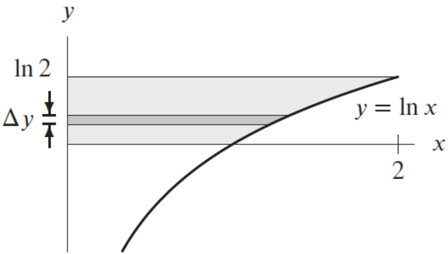
§8.1-8.2, 7.6

- *(§8.1-8.2)* Volumes by discs and washers. **No volumes of known cross-section. No arc length.**
  - Questions like: §8.2 #1-17 (odds), 41-45 (odds), 51
- *(§7.6)* Improper integrals: infinite bounds vs. singularities in the integrand.
  - ⋆ Questions like: §7.6 #5-39 (odds)

Example (§8.2 #18)

The region bounded by $y = \ln x$, $x = 0$, $y = \ln 2$, and $y = 0$ is rotated around the y-axis. Write, then evaluate, an integral giving the volume.

Soln:



Graphic: Region bounded by $y = \ln x$, $x = 0$, $y = \ln 2$, and $y = 0$.

Since the region is rotated around the y-axis, we write the integral in terms of y.

The x-coordinate of the curve $y = \ln x$ gives the radius of one disc. Solve for x: $y = \ln x \implies x = e^y$.

The bounds on y are from 0 to $\ln 2$. The volume is

$$\begin{aligned}\int_0^{\ln 2} \pi (e^y)^2 \, dy &= \pi \int_0^{\ln 2} e^{2y} \, dy \\ &= \frac{\pi}{2} e^{2y} \Big|_0^{\ln 2} = \frac{\pi}{2}\left(e^{2\ln 2} - e^{2(0)}\right) \\ &= \frac{\pi}{2}\left(\left(e^{\ln 2}\right)^2 - 1\right) \\ &= \frac{3}{2}\pi.\end{aligned}$$

Example (§7.6 #24)

Calculate the integral $\int_{-1}^{1} \frac{dt}{\sqrt{t+1}}$ if it converges.

Soln: There is a singularity at $t = -1$. We use the placeholder variable a and take the limit as $a \to -1$ from the right.

$$\int_{-1}^{1} \frac{dt}{\sqrt{t+1}} = \lim_{a \to -1^+} \int_{a}^{1} \frac{dt}{\sqrt{t+1}}$$

To evaluate the integral, use a linear substitution $w = t + 1 \implies dw = dt$. The bounds become $w(a) = a + 1$ and $w(1) = 1 + 1 = 2$.

$$\begin{aligned}
\lim_{a\to -1^+}\int_a^1 \frac{dt}{\sqrt{t+1}} &= \lim_{a\to -1^+}\int_{a+1}^2 \frac{dw}{\sqrt{w}} \\
&= 2\lim_{a\to -1^+}\sqrt{w}\,\Big|_{a+1}^{2} \\
&= 2\lim_{a\to -1^+}(\sqrt{2}-\sqrt{a+1}) \\
&= 2(\sqrt{2}-\sqrt{0}) = 2\sqrt{2}
\end{aligned}$$