Calculus I (Math 2554)

Spring 2016

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University of Arkansas

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The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

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Tips for Success

- Attend class every day. Participate in math discussions. Do the lecture-cises fully, not just on a scratch paper.
- Don't get behind on MLP homeworks. Stay on top of the book problems.
- Find a study partner(s) to meet with on a regular basis. Don't be afraid to seek further assistance (tutoring, office hours, etc.) if you are struggling.
- high school calculus $\neq$ college calculus
- REMEMBER... THE TERM STARTS TODAY! SO DOES THE EVENTUAL EARNING OF YOUR FINAL GRADE!!!

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Wed 20 Jan

Welcome to Cal I!

- comp.uark.edu/~ashleykw/Cal1Spring2016/cal1spr16.html
  Course website. All information is here, including a link to MLP, lecture slides, administrative information, etc. You should have already seen the **syllabus** by now.
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Wed 20 Jan (cont.)

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§2.1 The Idea of Limits

Question

How would you define, and then differentiate between, the following pairs of terms?

- instantaneous velocity vs. average velocity?
- tangent line vs. secant line?

(Recall: What is a tangent line and what is a secant line?)

Example

An object is launched into the air. Its position s (in feet) at any time t (in seconds) is given by the equation:

$$s(t) = -4.9t^2 + 30t + 20.$$

(a) Compute the average velocity of the object over the following time intervals: $[1, 3]$, $[1, 2]$, $[1, 1.5]$

(b) As your interval gets shorter, what do you notice about the average velocities? What do you think would happen if we computed the average velocity of the object over the interval $[1, 1.2]$? $[1, 1.1]$? $[1, 1.05]$?

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Example, cont.

An object is launched into the air. Its position s (in feet) at any time t (in seconds) is given by the equation:

$$s(t) = -4.9t^2 + 30t + 20.$$

(c) How could you use the average velocities to estimate the instantaneous velocity at $t = 1$?

(d) What do the average velocities you computed in 1. represent on the graph of $s(t)$?

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Question

What happens to the relationship between instantaneous velocity and average velocity as the time interval gets shorter?

Answer: The instantaneous velocity at $t = 1$ is the limit of the average velocities as t approaches 1.

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Question

What about the relationship between the secant lines and the tangent lines as the time interval gets shorter?

Answer: The slope of the tangent line at $(1, 45.1 = s(1))$ is the limit of the slopes of the secant lines as t approaches 1.

2.1 Book Problems

1-3, 7-13, 15, 21, 25, 27, 29

Even though book problems aren't turned in, they're a very good way to study for quizzes and tests (wink wink wink).

§2.2 Definition of Limits

Question

- Based on your everyday experiences, how would you define a "limit"?
- Based on your mathematical experiences, how would you define a "limit"?
- How do your definitions above compare or differ?

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Fri 22 Jan

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Fri 22 Jan (cont.)

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- For old Calculus materials, see the parent page `comp.uark.edu/~ashleykw` and look for links under "Previous Semesters".
- Next week: Attendance using clickers.

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Definition of a Limit of a Function

Definition (limit)

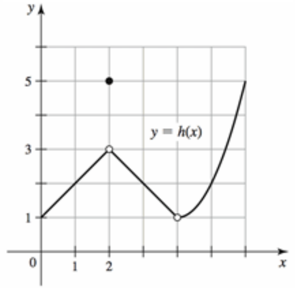
Suppose the function f is defined for all x near a, except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **equals** L.

Determining Limits from a Graph

Exercise



Determine the following:

(a) $h(1)$

(b) $h(2)$

(c) $h(4)$

(d) $\lim_{x\to 2} h(x)$

(e) $\lim_{x\to 4} h(x)$

(f) $\lim_{x\to 1} h(x)$

Question

Does $\lim_{x \to a} f(x)$ always equal $f(a)$?

(Hint: Look at the example from the previous slide!)

Determining Limits from a Table

Exercise

Suppose $f(x) = \dfrac{x^2 + x - 20}{x - 4}$.

(a) Create a table of values of $f(x)$ when

$$x = 3.9, 3.99, 3.999, \text{ and}$$
$$x = 4.1, 4.01, 4.001$$

(b) What can you conjecture about $\lim_{x \to 4} f(x)$?

One-Sided Limits

Up to this point we have been working with two-sided limits; however, for some functions it makes sense to examine one-sided limits.

Notice how in the previous example we could approach $f(x)$ from both sides as x approaches a, i.e., when $x > a$ and when $x < a$.

Definition (right-hand limit)

Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the right equals** L.

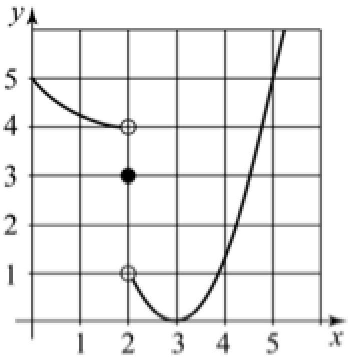
Definition (left-hand limit)

Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$

and say **the limit of** $f(x)$ **as** x **approaches** a **from the left equals** L.

Exercise



Determine the following:

(a) $g(2)$

(b) $\lim_{x\to 2^+} g(x)$

(c) $\lim_{x\to 2^-} g(x)$

(d) $\lim_{x\to 2} g(x)$

Relationship Between One- and Two-Sided Limits

Theorem

If f is defined for all x near a except possibly at a, then $\lim_{x\to a} f(x) = L$ if and only if both $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

In other words, the only way for a two-sided limit to exist is if the one-sided limits equal the same number (L).

2.2 Book Problems

1-4, 7, 9, 11, 13, 19, 23, 29, 31

1 19-22 January
- Wednesday 20 January

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§2.3 Techniques for Computing Limits
- Limit Laws
- Limits of Polynomials and Rational Functions

2 25-29 January

3 1-5 February

4 8-12 February

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§2.3 Techniques for Computing Limits

Exercise

Given the function $f(x) = 4x + 7$, find $\lim_{x\to -2} f(x)$

(a) graphically;

(b) numerically (i.e., using a table of values near -2)

(c) via a direct computation method of your choosing.

Compare and contrast the methods in this exercise – which is the most favorable?

This section provides various laws and techniques for determining limits. These constitute **analytical** methods of finding limits. The following is an example of a very useful limit law:

Limits of Linear Functions: Let a, b, and m be real numbers. For linear functions $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b.$$

This rule says we if $f(x)$ is a linear function, then in taking the limit as $x \to a$, we can just plug in the a for x.

IMPORTANT! Using a table or a graph to compute limits, as in the previous sections, can be helpful. However, "analytical" does not include those techniques.

Limit Laws

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

---

1. **Sum:** $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. **Difference:**
$\lim_{x\to a} (f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

In other words, if we are taking a limit of two things added together or subtracted, then we can first compute each of their individual limits one at a time.

Limit Laws, cont.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

---

3. **Constant Multiple:** $\lim_{x\to a} (cf(x)) = c\left(\lim_{x\to a} f(x)\right)$

4. **Product:** $\lim_{x\to a} (f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$

The same is true for products. If one of the factors is a constant, we can just bring it outside the limit. In fact, a constant is its own limit.

Limit Laws, cont.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

---

5. **Quotient:** $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$

   (provided $\lim_{x\to a} g(x) \neq 0$)

Question

Why the caveat?

Limit Laws, cont.

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, c is a real number, and m, n are positive integers.

---

6. **Power:** $\lim_{x\to a} (f(x))^n = (\lim_{x\to a} f(x))^n$
7. **Fractional Power:** $\lim_{x\to a} (f(x))^{\frac{n}{m}} = (\lim_{x\to a} f(x))^{\frac{n}{m}}$

   (provided $f(x) \geq 0$ for x near a if m is even and $\frac{n}{m}$ is in lowest terms)

Question

Explain the caveat in **7.**

Limit Laws, cont.

Laws **1.-6.** hold for one-sided limits as well. But **7.** must be modified:

7. **Fractional Power (one-sided limits):**
   - $\lim_{x\to a^+} (f(x))^{\frac{n}{m}} = (\lim_{x\to a^+} f(x))^{\frac{n}{m}}$
     (provided $f(x) \geq 0$ for x near a with $x > a$, if m is even and $\frac{n}{m}$ is in lowest terms)
   - $\lim_{x\to a^-} (f(x))^{\frac{n}{m}} = (\lim_{x\to a^-} f(x))^{\frac{n}{m}}$
     (provided $f(x) \geq 0$ for x near a with $x < a$, if m is even and $\frac{n}{m}$ is in lowest terms)

Limits of Polynomials and Rational Functions

Assume that $p(x)$ and $q(x)$ are polynomials and a is a real number.

- **Polynomials:** $\lim_{x \to a} p(x) = p(a)$
- **Rational functions:** $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$

  (provided $q(a) \neq 0$)

For polynomials and rational functions we can plug in a to compute the limit, as long as we don't get zero in the denominator. Linear functions count as polynomials. A rational function is a "fraction" made of polynomials.

Exercise

Evaluate the following limits analytically.

1. $\lim_{x\to 1} \frac{4f(x)g(x)}{h(x)}$, given that

$$\lim_{x\to 1} f(x) = 5, \ \lim_{x\to 1} g(x) = -2, \text{ and } \lim_{x\to 1} h(x) = -4.$$

2. $\lim_{x\to 3} \frac{4x^2+3x-6}{2x-3}$

3. $\lim_{x\to 1^-} g(x)$ and $\lim_{x\to 1^+} g(x)$, given that

$$g(x) = \begin{cases} x^2 & \text{if } x \leq 1; \\ x+2 & \text{if } x > 1. \end{cases}$$

Mon 25 Jan

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Mon 25 Jan (cont.)

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- GET YOUR CLICKER
- Note: There is no Blackboard for this course.
- Stay on top of the MLP! First deadline is coming up. Don't wait till the last minute.
- MLP issues...
- Quiz 1 is due in drill tomorrow. See MLP for a copy.

Additional (Algebra) Techniques

When direct substitution (a.k.a. plugging in a) fails try using algebra:

- Factor and see if the denominator cancels out.

Example

$\lim_{t\to 2} \frac{3t^2-7t+2}{2-t}$

- Look for a common denominator.

Example

$\lim_{h\to 0} \frac{\frac{1}{5+h}-\frac{1}{5}}{h}$

Exercise

Evaluate $\lim_{s\to 3} \frac{\sqrt{3s+16}-5}{s-3}$.

Another Technique: Squeeze Theorem

This method for evaluating limits uses the relationship of functions with each other.

Theorem (Squeeze Theorem)

Assume $f(x) \leq g(x) \leq h(x)$ *for all values of* x *near* a*, except possibly at* a*, and suppose*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Then since g *is always between* f *and* h *for* x*-values close enough to* a*, we must have*

$$\lim_{x\to a} g(x) = L.$$

Example

(a) Draw a graph of the inequality

$$-|x| \leq x^2 \ln\left(x^2\right) \leq |x|.$$

(b) Compute $\lim_{x\to 0} x^2 \ln\left(x^2\right)$.

2.3 Book Problems

12-30 (every 3rd problem), 33, 39-51 (odds), 55, 57, 61-67 (odds)

In general, review your algebra techniques, since they can save you some headache.

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§2.4 Infinite Limits

We have examined a number of laws and methods to evaluate limits.

Question

Consider the following limit:

$$\lim_{x\to 0} \frac{1}{x}$$

How would you evaluate this limit?

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In the next two sections, we examine limit scenarios involving infinity. The two situations are:

- **Infinite limits:** as x (i.e., the independent variable) approaches a finite number, y (i.e., the dependent variable) becomes arbitrarily large or small

  $$\text{looks like: } \lim_{x\to\text{number}} f(x) = \pm\infty$$

- **Limits at infinity:** as x approaches an arbitrarily large or small number, y approaches a finite number

  $$\text{looks like: } \lim_{x\to\pm\infty} f(x) = \text{number}$$

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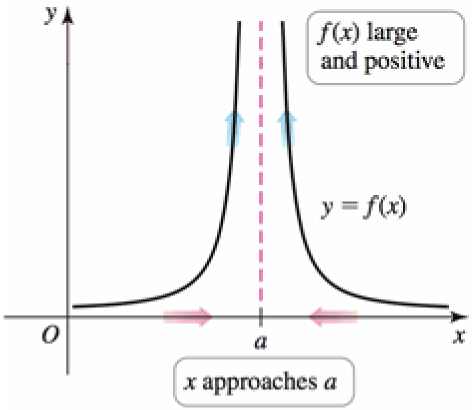
Definition of Infinite Limits

Definition (positively infinite limit)

Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = \infty$$

and say **the limit of $f(x)$ as x approaches a is infinity**.



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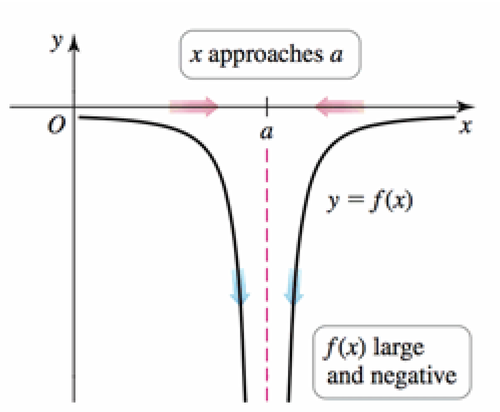
Definition (negatively infinite limit)

Suppose f is defined for all x near a. If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = -\infty$$

and say **the limit of $f(x)$ as x approaches a is negative infinity**.



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The definitions work for one-sided limits, too.

Exercise

Using a graph and a table of values, given $f(x) = \dfrac{1}{x^2 - x}$, determine:

(a) $\displaystyle\lim_{x \to 0^+} f(x)$

(b) $\displaystyle\lim_{x \to 0^-} f(x)$

(c) $\displaystyle\lim_{x \to 1^+} f(x)$

(d) $\displaystyle\lim_{x \to 1^-} f(x)$

Wed 27 Jan

- GET YOUR CLICKER. Starting next week, no attendance sheet, clickers only.
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Definition of Vertical Asymptote

Definition

Suppose a function f satisfies at least one of the following:

- $\lim_{x \to a} f(x) = \pm\infty$,
- $\lim_{x \to a^+} f(x) = \pm\infty$
- $\lim_{x \to a^-} f(x) = \pm\infty$

Then the line $x = a$ is called a **vertical asymptote** of f.

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Exercise

Given $f(x) = \frac{3x-4}{x+1}$, determine, analytically (meaning using "number sense" and without a table or a graph),

(a) $\lim_{x \to -1^+} f(x)$

(b) $\lim_{x \to -1^-} f(x)$

Summary Statements

Here is a common way you can summarize your solutions involving limits:

"Since the numerator approaches ($\#$) and the denominator approaches 0, and is (positive/negative), and since (analyze signs here), (insert limit problem)=($+\infty$ / $-\infty$)."

Remember to check for factoring –

Exercise

(a) What is/are the vertical asymptotes of

$$f(x) = \frac{3x^2 - 48}{x + 4}?$$

(b) What is $\lim_{x \to -4} f(x)$? Does that correspond to your earlier answer?

2.4 Book Problems

7-10, 15, 17-23, 31-34, 44-45

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§2.5 Limits at Infinity

Limits at infinity determine what is called the **end behavior** of a function.

Exercise

(a) Evaluate the following functions at the points $x = \pm 100, \pm 1000, \pm 10000$;

$$f(x) = \frac{4x^2 + 3x - 2}{x^2 + 2} \qquad g(x) = -2 + \frac{\cos x}{\sqrt[3]{x}}$$

(b) What is your conjecture about $\lim_{x \to \infty} f(x)$? $\lim_{x \to -\infty} f(x)$? $\lim_{x \to -\infty} g(x)$? $\lim_{x \to \infty} g(x)$?

Horizontal Asymptotes

Definition

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x\to\infty} f(x) = L.$$

The line $y = L$ is a **horizontal asymptote** of f.

The limit at negative infinity, $\lim_{x\to-\infty} f(x) = M$, is defined analogously and in this case, the horizontal asymptote is $y = M$.

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Infinite Limits at Infinity

Question

Is it possible for a limit to be both an infinite limit and a limit at infinity? (Yes.)

If $f(x)$ becomes arbitrarily large as x becomes arbitrarily large, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

(The limits $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$ are defined similarly.)

Powers and Polynomials: Let n be a positive integer and let $p(x)$ be a polynomial.

- $n =$ even number: $\lim\limits_{x\to\pm\infty} x^n = \infty$
- $n =$ odd number: $\lim\limits_{x\to\infty} x^n = \infty$ and $\lim\limits_{x\to-\infty} x^n = -\infty$

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- (again, assuming n is positive)

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0$$

- For a polynomial, only look at the term with the highest exponent:

$$\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} (\text{constant}) \cdot x^n$$

The constant is called the **leading coefficient**, $\text{lc}(p)$. The highest exponent that appears in the polynomial is called the **degree**, $\deg(p)$.

Rational Functions: Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function.

- If $\deg(p) < \deg(q)$, i.e., the numerator has the smaller degree, then
$$\lim_{x\to\pm\infty} f(x) = 0$$
and $y = 0$ is a horizontal asymptote of f.

- If $\deg(p) = \deg(q)$, i.e., numerator and denominator have the same degree, then
$$\lim_{x\to\pm\infty} f(x) = \frac{\mathsf{lc}(p)}{\mathsf{lc}(q)}$$
and $y = \frac{\mathsf{lc}(p)}{\mathsf{lc}(q)}$ is a horizontal asymptote of f.

- If $\deg(p) > \deg(q)$, (numerator has the bigger degree) then

$$\lim_{x\to\pm\infty} f(x) = \infty \quad \text{or} \quad -\infty$$

  and f has no horizontal asymptote.

- Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeroes of q.

  (This is why it is a good idea to check for factoring and cancelling first!)

When evaluating limits at infinity for rational functions, it is not enough to use the previous rule to show the limit analytically.

To evaluate these limits, we divide both numerator and denominator by x^n, where n is the degree of the polynomial in the denominator.

Fri 29 Jan

- Today: You MUST sign in if your name is highlighted. Everyone must click in, if possible.
- GET YOUR CLICKER NOW. Starting next week, no attendance sheet, clickers only.
- There is no Blackboard for this course.
- Stay on top of the MLP! First deadline is SUNDAY. Don't wait till the last minute.
- EXAM 1 is in two weeks, covers up to §3.1 (see the semester schedule of material on the course webpage). You must attend your own lecture on exam day.

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Exercise

Determine the end behavior of the following functions (in other words, compute both limits, as $x \to \pm\infty$, for each of the functions):

1. $f(x) = \dfrac{x+1}{2x^2-3}$
2. $g(x) = \dfrac{4x^3-3x}{2x^3+5x^2+x+2}$
3. $h(x) = \dfrac{6x^4-1}{4x^3+3x^2+2x+1}$

Algebraic and Transcendental Functions

Example

Determine the end behavior of the following functions.

1. $f(x) = \dfrac{4x^3}{2x^3 + \sqrt{9x^6 + 15x^4}}$ (radical signs appear)
2. $g(x) = \cos x$ (trig)
3. $h(x) = e^x$ (exponential)

Mon 1 Feb

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Exercise

What are the vertical and horizontal asymptotes of $f(x) = \frac{x^2}{2x+1}$?

2.5 Book Problems

9-14, 15-33 (odds), 41-49 (odds), 53-59 (odds), 67

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

§2.6 Continuity

Informally, a function f is "continuous at $x = a$" means for x-values anywhere close enough to a the graph can be drawn without lifting a pencil. In other words, no holes, breaks, asymptotes, etc.

Definition

A function f is **continuous** at a means

$$\lim_{x \to a} f(x) = f(a).$$

If f is not continuous at a, then a is a **point of discontinuity**.

Continuity Checklist

In order to claim something is continuous, you must verify all three:

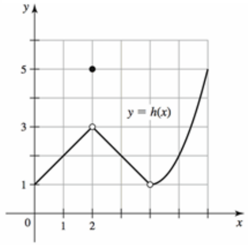
1. $f(a)$ is defined (i.e., a is in the domain of f – no holes, asymptotes).
2. $\lim_{x \to a} f(x)$ exists. You must check both sides and make sure they equal the same number.
3. $\lim_{x \to a} f(x) = f(a)$ (i.e., the value of f equals the limit of f at a).

Question

What is an example of a function that satisfies this condition?

Example

- Where are the points of discontinuity of the function below?
- Which aspects of the checklist fail?



recall (Continuity Checklist):

1. function is defined
2. the two-sided limit exists
3. 2. = 1.

Continuity Rules

If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$, provided $g(a) \neq 0$
6. $[f(x)]^n$

From the rules above, we can deduce:

1. Polynomials are continuous for all $x = a$.
2. Rational functions are continuous at all $x = a$ except for the points where the denominator is zero.
3. If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Continuity on an Interval

Consider the cases where f is not defined past a certain point.

Definition

A function f is **continuous from the left** (or **left-continuous**) at a means

$$\lim_{x \to a^-} f(x) = f(a);$$

a function f is **continuous from the right** (or **right-continuous**) at a means

$$\lim_{x \to a^+} f(x) = f(a).$$

Definition

A function f is **continuous on an interval** I means it is continuous at all points of I.

Notation: Intervals are usually written

$$[a, b]\,,\ (a, b]\,,\ [a, b)\,,\ \text{or}\ (a, b)\,.$$

When I contains its endpoints, "continuity on I" means continuous from the right or left at the endpoints.

Wed 3 Feb

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Example

Let $f(x) = \begin{cases} x^3 + 4x + 1 & \text{if } x \leq 0 \\ 2x^3 & \text{if } x > 0. \end{cases}$

1. Use the continuity checklist to show that f is not continuous at 0.
2. Is f continuous from the left or right at 0?
3. State the interval(s) of continuity.

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Continuity of Functions with Roots

(assuming m and n are positive integers and $\frac{n}{m}$ is in lowest terms)

- If m is odd, then $[f(x)]^{\frac{n}{m}}$ is continuous at all points at which f is continuous.
- If m is even, then $[f(x)]^{\frac{n}{m}}$ is continuous at all points a at which f is continuous and $f(a) \geq 0$.

Question

Where is $f(x) = \sqrt[4]{4 - x^2}$ continuous?

Continuity of Transcendental Functions

Trig Functions: The basic trig functions are all continuous at all points IN THEIR DOMAIN. Note there are points of discontinuity where the functions are not defined – for example, $\tan x$ has asymptotes everywhere that $\cos x = 0$.

Exponential Functions: The exponential functions b^x and e^x are continuous on all points of their domains.

Inverse Functions: If a continuous function f has an inverse on an interval I (meaning if $x \in I$ then $f^{-1}(y)$ passes the vertical line test), then its inverse f^{-1} is continuous on the interval J, which is defined as all the numbers $f(x)$, given x is in I.

Intermediate Value Theorem (IVT)

Theorem (Intermediate Value Theorem)

Suppose f is continuous on the interval $[a, b]$ and L is a number satisfying

$$f(a) < L < f(b) \quad \textit{or} \quad f(b) < L < f(a).$$

Then there is at least one number $c \in (a, b)$, i.e., $a < c < b$, satisfying

$$f(c) = L.$$

Example

Let $f(x) = -x^5 - 4x^2 + 2\sqrt{x} + 5$. Use IVT to show that $f(x) = 0$ has a solution in the interval $(0, 3)$.

Fri 5 Feb

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- EXAM 1 is one week from today. Covers up to §3.1 (see the semester schedule of material on the course webpage). You must attend your own lecture on exam day. CEA: Register with the CEA office for a time on 12 Feb, as close to your normal lecture time as possible.
- Look at old Wheeler exams to study. `comp.uark.edu/~ashleykw`

Exercise

Which of the following functions is continuous for all real values of x?

(A) $f(x) = \frac{x^2}{2x+1}$

(B) $g(x) = \sqrt{3x^2 - 2}$

(C) $h(x) = \frac{5x}{|x^8-1|}$

(D) $j(x) = \frac{5x}{x^8+1}$

2.6 Book Problems

9-25 (odds), 35-45 (odds), 59, 61, 63, 83, 85

1. 19-22 January
2. 25-29 January
3. 1-5 February
   - **Monday 1 February**
   - Book Problems

§2.6 Continuity
- Continuity Checklist
- Continuity Rules
- Continuity on an Interval
- **Wednesday 3 February**
- Continuity of Functions with Roots
- Continuity of Transcendental Functions
- Intermediate Value Theorem (IVT)
- **Friday 5 February**
- Book Problems

§2.7 Precise Definitions of Limits
- ϵ and δ
- Seeing ϵs and δs on a Graph
- Finding a Symmetric Interval

4. 8-12 February

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

§2.7 Precise Definitions of Limits

So far in our dealings with limits, we have used informal terms such as "sufficiently close" and "arbitrarily large". Now we will formalize what these terms mean mathematically.

Recall: $|f(x) - L|$ and $|x - a|$ refer to the distances between $f(x)$ and L and between x and a.

Also, recall that when we worked informally with limits, we wanted x to approach a, but not necessarily equal a. Likewise, we wanted f to get arbitrarily close to L, but not necessarily equal L.

Definition

Assume that $f(x)$ exists for all x in some open interval (open means: neither of the endpoints not included) containing a, except possibly at a. **"The limit of $f(x)$ as x approaches a is L"**, i.e.,

$$\lim_{x \to a} f(x) = L,$$

means **for any $\epsilon > 0$ there exists $\delta > 0$ such that**

$$|f(x) - L| < \epsilon \quad \textbf{whenever} \quad 0 < |x - a| < \delta.$$

Question

Why $0 < |x - a|$ but not for $|f(x) - L|$?

ϵ and δ

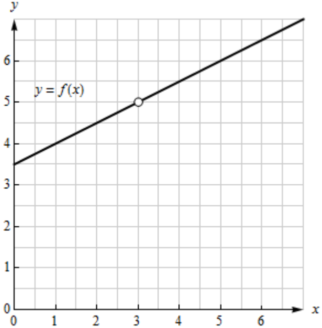
When we worked informally with limits, we saw $f(x)$ get closer and closer to L as x got closer and closer to a.

Question

If we want the distance between $f(x)$ and L to be less than 1, how close does x have to be to a? What if we want $|f(x) - L| < 0.5$? 0.5? 0.1? 0.01?

Seeing ϵs and δs on a Graph

Example



Using the graph, for each $\epsilon > 0$, determine a value of $\delta > 0$ to satisfy the statement

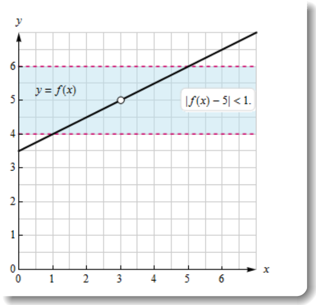
$$|f(x) - 5| < \epsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta.$$

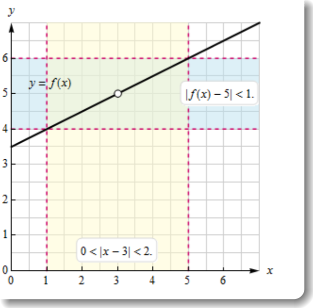
(a) $\epsilon = 1$

(b) $\epsilon = 0.5$.

Seeing ϵs and δs on a Graph, cont.

When $\epsilon = 1$:

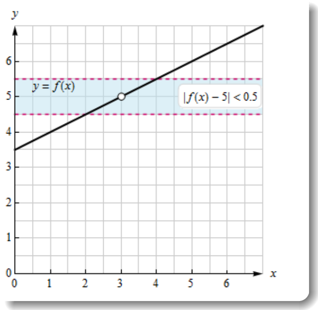


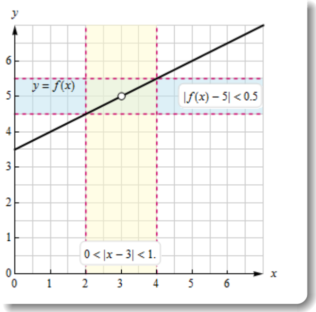


$\ldots\ \delta = 2$

Seeing ϵs and δs on a Graph, cont.

When $\epsilon = 0.5$:





$\ldots\ \delta = 1$

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The ϵs and δs give a way to visualize computing the limit, and prove it exists. As the ϵs get smaller and smaller, we want there to always be a δ. In this example,

$$\lim_{x \to 3} f(x) = 5.$$

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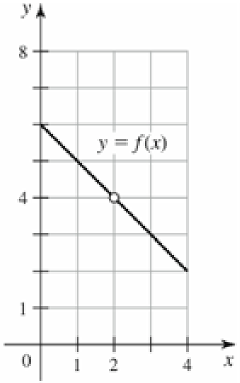
Exercise



Using the graph, for each $\epsilon > 0$, determine a value of $\delta > 0$ to satisfy the statement

$$|f(x) - 4| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

(a) $\epsilon = 1$

(b) $\epsilon = 0.5$.

Finding a Symmetric Interval

Question

When finding an interval $(a - \delta, a + \delta)$ around the point a, what happens if you compute two different δs?

Answer: To obtain a symmetric interval around a, use the smaller of the two δs as your distance around a.

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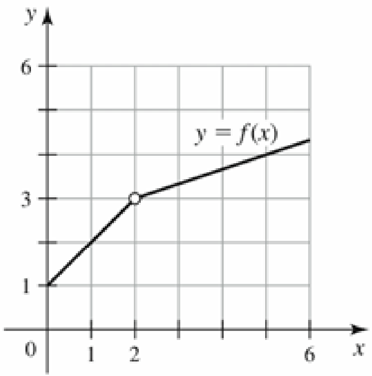
Exercise



The graph of $f(x)$ shows

$$\lim_{x \to 2} f(x) = 3.$$

For $\epsilon = 1$, find the corresponding value of $\delta > 0$ so that

$$|f(x) - 3| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

Mon 8 Feb

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  - Look at old Wheeler exams to study. `comp.uark.edu/~ashleykw`
  - Also look at Quiz and Drill solutions posted in MLP.
  - Do the book problems. Go to office hours or Calculus Corner to get feedback.

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Mon 8 Feb (cont.)

- Quizzes:
  - Include drill instructor and time.
  - Don't turn in the Quiz sheet with your work.
  - No quiz again until next week.
  - Drill Exercise Tues 16 Feb and Quiz 4 Thurs 18 Feb.

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Mon 8 Feb (cont.)

- Announcement:
  A student in this class requires a note-taker. If you are willing to upload your notes and plan to attend class on a REGULAR basis, please sign up via the CEA Online Services on the Center for Educational Access (CEA) website `http://cea.uark.edu`. On the CEA Online Services login screen, click on "Sign Up as a Note-taker". At the end of the semester you will receive verification of 48 community service hours OR a $50 gift card for providing class notes. All interested students are encouraged to sign up; preference may be given to volunteers seeking community service in an effort engage U of A students in community service opportunities. Please contact the Center for Educational Access at ceanotes@uark.edu if you have any questions.

Exercise

Let $f(x) = x^2 - 4$. For $\epsilon = 1$, find a value for $\delta > 0$ so that

$$|f(x) - 12| < \epsilon \quad \text{whenever} \quad 0 < |x - 4| < \delta.$$

In this example, $\lim_{x \to 4} f(x) = 12$.

2.7 Book Problems

1-7, 9-18

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§3.1 Introducing the Derivative

Recall from Ch 2: We said that the slope of the tangent line at a point is the limit of the slopes of the secant lines as the points get closer and closer.

- slope of secant line: $\dfrac{f(x)-f(a)}{x-a}$ (average rate of change)
- slope of tangent line: $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ (instantaneous rate of change)

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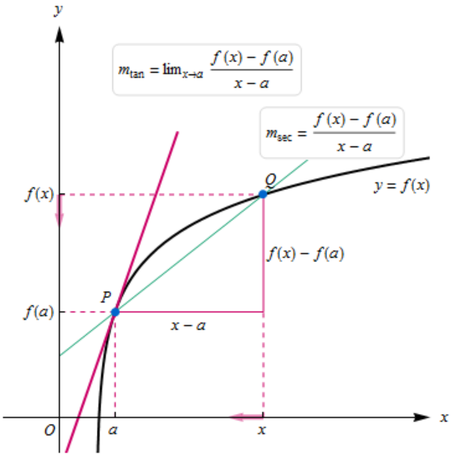

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Exercise

Use the relationship between secant lines and tangent lines, specifically the slope of the tangent line, to find the equation of a line tangent to the curve $f(x) = x^2 + 2x + 2$ at the point $P = (1, 5)$.

In the preceding exercise, we considered two points

$$P = (a, f(a)) \quad \text{and} \quad Q = (x, f(x))$$

that were getting closer and closer together.

Instead of looking at the points approaching one another, we can also view this as the distance h between the points approaching 0. For

$$P = (a, f(a)) \quad \text{and} \quad Q = (a+h, f(a+h)),$$

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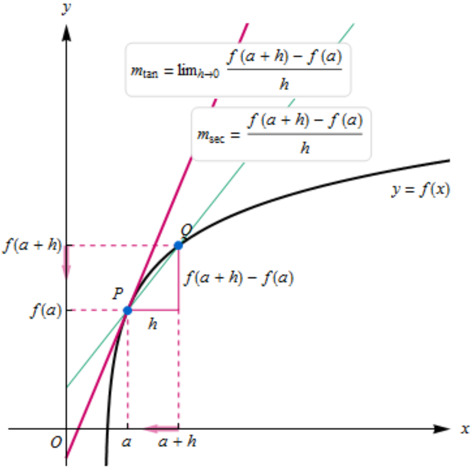
- slope of secant line:

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

- slope of tangent line:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$



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Exercise

Find the equation of a line tangent to the curve $f(x) = x^2 + 2x + 2$ at the point $P = (2, 10)$.

Derivative Defined as a Function

The slope of the tangent line for the function f is itself a function of x (in other words, there is an expression where we can plug in any value $x = a$ and get the derivative at that point), called the derivative of f.

Definition

The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. If $f'(x)$ exists, we say f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

Exercise

Use the definition of the derivative to find the derivative of the function $f(x) = x^2 + 2x + 2$.

Leibniz Notation

A standard notation for change involves the Greek letter Δ.

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}.$$

Apply the limit:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Other Notation

The following are alternative ways of writing $f'(x)$ (i.e., the derivative as a function of x):

$$\frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx}(f(x)) \qquad D_x(f(x)) \qquad y'(x)$$

The following are ways to notate the derivative of f evaluated at $x = a$:

$$f'(a) \qquad y'(a) \qquad \left.\frac{df}{dx}\right|_{x=a} \qquad \left.\frac{dy}{dx}\right|_{x=a}$$

Wed 10 Feb

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Wed 10 Feb (cont.)

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Wed 10 Feb (cont.)

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Question

Do the words “derive” and “differentiate” mean the same thing?

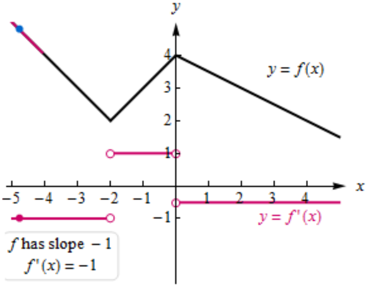
Graphing the Derivative

The graph of the derivative is the graph of the collection of slopes of tangent lines of a graph. If you just have a graph (without an equation for the graph), the best you can do is approximate the graph of the derivative.

Example

Simple checklist:

1. Note where $f'(x) = 0$.
2. Note where $f'(x) > 0$.
   (What does this look like?)
3. Note where $f'(x) < 0$.
   (What does this look like?)



Differentiability vs. Continuity

Key points about the relationship between differentiability and continuity:

- If f is differentiable at a, then f is continuous at a.
- If f is not continuous at a, then f is not differentiable at a.
- f can be continuous at a, but not differentiable at a.

A function f is not differentiable at a if at least one of the following conditions holds:

1. f is not continuous at a.
2. f has a corner at a.

Question

Why does this make f not differentiable?

3. f has a vertical tangent at a.

Question

Why does this make f not differentiable?

3.1 Book Problems

9-45 (odds), 49-53 (odds)

- **NOTE:** You do not know any rules for differentiation yet (e.g., Power Rule, Chain Rule, etc.) In this section, you are strictly using the definition of the derivative and the definition of slope of tangent lines we have derived.

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Exam #1 Review

- §2.1 The Idea of Limits
  - Understand the relationship between average velocity & instantaneous velocity, and secant and tangent lines
  - Be able to compute average velocities and use the idea of a limit to approximate instantaneous velocities
  - Be able to compute slopes of secant lines and use the idea of a limit to approximate the slope of the tangent line
- §2.2 Definitions of Limits
  - Know the definition of a limit
  - Be able to use a graph of a table to determine a limit
  - Know the relationship between one- and two-sided limits

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Exam #1 Review (cont.)

- §2.3 Techniques for Computing Limits
  - Know and be able to compute limits using analytical methods (e.g., limit laws, additional techniques)
  - Know the Squeeze Theorem and be able to use it to determine limits

Example

Evaluate $\lim_{x\to 0} x \sin \frac{1}{x}$.

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Exam #1 Review (cont.)

- §2.4 Infinite Limits
  - Be able to use a graph, a table, or analytical methods to determine infinite limits
  - Know the definition of a vertical asymptote and be able to determine whether a function has vertical asymptotes
- §2.5 Limits at Infinity
  - Be able to find limits at infinity and horizontal asymptotes
  - Know how to compute the limits at infinity of rational functions

Exam #1 Review (cont.)

Example

Determine the end behavior of $f(x)$. If there is a horizontal asymptote, then say so. Next, identify any vertical asymptotes. If $x = a$ is a vertical asymptote, then evaluate $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$.

$$f(x) = \frac{2x^3 + 10x^2 + 12x}{x^3 + 2x^2}$$

Exam #1 Review (cont.)

- §2.6 Continuity
  - Know the definition of continuity and be able to apply the continuity checklist
  - Be able to determine the continuity of a function (including those with roots) on an interval
  - Be able to apply the Intermediate Value Theorem to a function

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

Exam #1 Review (cont.)

Example

Determine the value for a that will make $f(x)$ continuous.

$$f(x) = \begin{cases} \frac{x^2+3x+2}{x+1} & x \neq -1 \\ a & x = -1 \end{cases}$$

Example

Show that $f(x) = 2$ has a solution on the interval $(-1, 1)$, with

$$f(x) = 2x^3 + x.$$

Exam #1 Review (cont.)

Exercise

What value of k makes

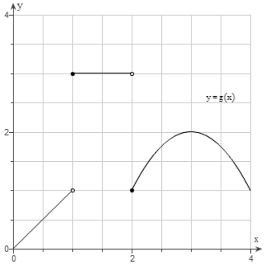
$$f(x) = \begin{cases} \frac{\sqrt{2x-5}-\sqrt{x+7}}{x-2} & x \neq 2 \\ k & x = 2 \end{cases}$$

continuous everywhere?

- $\oint$2.7 Precise Definition of Limits
  - Understand the δ, ϵ relationship for limits
  - Be able to use a graph or analytical methods to find a value for $\delta > 0$ given an $\epsilon > 0$ (including finding symmetric intervals)

Exam #1 Review (cont.)

Example



Use the graph to find the appropriate δ.

(a) $|g(x) - 2| < \frac{1}{2}$ whenever

$$0 < |x - 3| < \delta$$

(b) $|g(x) - 1| < \frac{3}{2}$ whenever

$$0 < |x - 2| < \delta$$

In this example, the two-sided limits at $x = 1$ and $x = 2$ do not exist.

Exam #1 Review (cont.)

- §3.1 Introducing the Derivative
    - Know the definition of a derivative and be able to use this definition to calculate the derivative of a given function
    - Be able to determine the equation of a line tangent to the graph of a function at a given point
    - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

Exam #1 Review (cont.)

Example

(a) Use the limit definition of the derivative to find an equation for the line tangent to $f(x)$ at a, where

$$f(x) = \frac{1}{x}; \quad a = -5.$$

(b) Using the same $f(x)$ from part (a), find a formula for $f'(x)$ (using the limit definition).

(c) Plug -5 into your answer for (b) and make sure it matches your answer for (a).

Other Study Tips

- Brush up on algebra, especially radicals.
- When in doubt, show steps. Defer to class notes and old exams to get an idea of what's expected.
- You will be punished for wrong notation; e.g., the limit symbol.
- Read the question! Several students always lose points because they didn't answer the question or they didn't follow directions.
- Do the book problems.
- Budget your time. You don't have to do the problems in order. Do the easier ones first.

Part 2. Derivatives

Part 2. Derivatives (cont.)

- **Wed 24 February**

§3.5 Derivatives of Trigonometric Functions

- Derivatives of Sine and Cosine Functions
- Trig Identities You Should Know
- Derivatives of Other Trig Functions
- Higher-Order Trig Derivatives
- Book Problems

§3.6 Derivatives as Rates of Change

- Position and Velocity
- Speed and Acceleration
- Growth Models
- Average and Marginal Cost
- Book Problems

7 29 Feb – 4 March

- **Wednesday 2 March**

§3.7 The Chain Rule

- Version 1 of the Chain Rule
- Guidelines for Using the Chain Rule
- Version 2 of the Chain Rule
- Chain Rule for Powers
- Composition of 3 or More Functions
- Book Problems

The base for these slides was done by Dr. Shannon Dingman, later encoded into LATEX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

Part 2. Derivatives (cont.)

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Mon 15 Feb

- Expect Exam back on Thursday.
- Quizzes:
  - Include drill instructor and time.
  - Don't turn in the Quiz sheet with your work.
  - Drill Exercise Tues 16 Feb and Quiz 4 Thurs 18 Feb.

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Mon 15 Feb (cont.)

- Announcement:
  A student in this class requires a note-taker. If you are willing to upload your notes and plan to attend class on a REGULAR basis, please sign up via the CEA Online Services on the Center for Educational Access (CEA) website `http://cea.uark.edu`. On the CEA Online Services login screen, click on "Sign Up as a Note-taker". At the end of the semester you will receive verification of 48 community service hours OR a $50 gift card for providing class notes. All interested students are encouraged to sign up; preference may be given to volunteers seeking community service in an effort engage U of A students in community service opportunities. Please contact the Center for Educational Access at ceanotes@uark.edu if you have any questions.

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5 15-19 February
- Monday 15 February

§3.2 Graphing the Derivative
- Book Problems

§3.3 Rules of Differentiation
- Constant Functions
- Power Rule
- Constant Multiple Rule
- Sum Rule
- Exponential Functions
- Wednesday 17 February
- Higher-Order Derivatives
- Book Problems

§3.4 The Product and Quotient Rules
- Product Rule
- Derivation of the Product Rule
- Derivation of the Quotient Rule
- Quotient Rule

6 22-26 February

7 29 Feb – 4 March

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§3.2 Graphing the Derivative

Recall: The graph of the derivative is essentially the graph of the collection of slopes of the tangent lines of a graph. If you just have a graph (without an equation for the graph), the best you can do is approximate the graph of the derivative.

Simple Checklist:

1. Note where $f'(x) = 0$.
2. Note where $f'(x) > 0$.

Question

What does this look like?

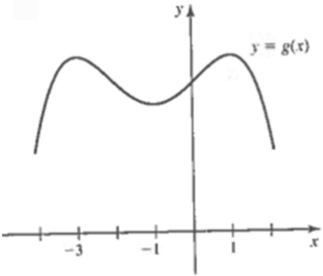
3. Note where $f'(x) < 0$.

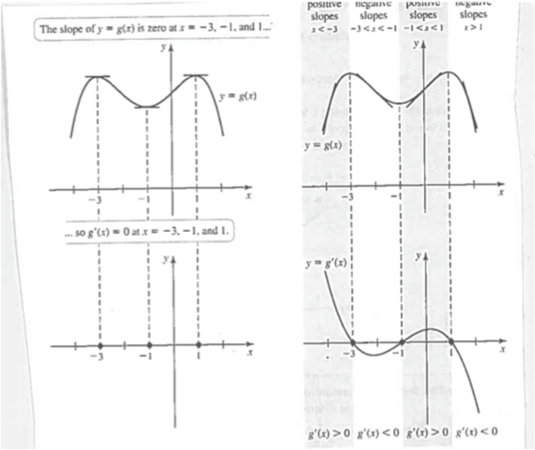
Question

What does this look like?

Example

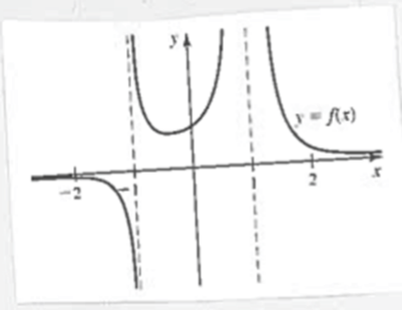
Given the graph of $g(x)$, sketch the graph of $g'(x)$.

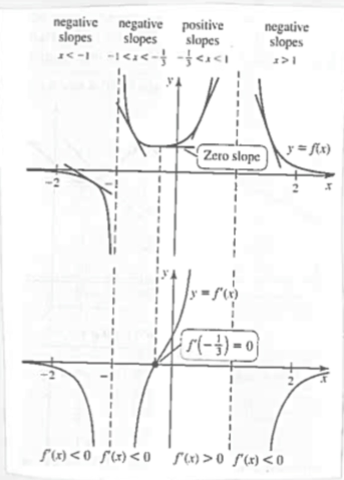



The slope of $y = g(x)$ is zero at $x = -3, -1$, and 1...
$y = g(x)$
... so $g'(x) = 0$ at $x = -3, -1$, and 1.
positive slopes $x < -3$
negative slopes $-3 < x < -1$
positive slopes $-1 < x < 1$
negative slopes $x > 1$
$y = g'(x)$
$g'(x) > 0$
$g'(x) < 0$
$g'(x) > 0$
$g'(x) < 0$

Example (With Asymptotes)

Given the graph of $f(x)$, sketch the graph of $f'(x)$.





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Recall the relationship between differentiability and continuity.

Exercise

If a function g is not continuous at $x = a$, then g

A. must be undefined at $x = a$.

B. is not differentiable at $x = a$.

C. has an asymptote at $x = a$.

D. all of the above.

E. A. and B. only.

3.2 Book Problems

5-14

§3.3 Rules of Differentiation

Recall the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

(as a function of x, i.e., a formula).
And, for any particular point a, we have

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Constant Functions

The constant function $f(x) = c$ is a horizontal line with a slope of 0 at every point. This is consistent with the definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{c - c}{h} \\ &= \lim_{h\to 0} 0 = 0. \end{aligned}$$

Therefore, for constant functions, $\frac{d}{dx}c = 0$.

Power Rule

Fact: For any positive integer n, we can factor

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

For example, when $n = 2$, we get

$$x^2 - a^2 = (x-a)(x+a),$$

which is the difference of squares formula.

Power Rule, cont.

Suppose $f(x) = x^n$ where n is a positive integer. Then at a point a,

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\ &= (a^{n-1} + a^{n-2}\cdot a + \cdots + a\cdot a^{n-2} + a^{n-1}) = na^{n-1}. \end{aligned}$$

Using the formula for the derivative as a function of x, one can show $\frac{d}{dx}(x^n) = nx^{n-1}$.

Constant Multiple Rule

Consider a function of the form $cf(x)$, where c is a constant. Just like with limits, we can factor out the constant:

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\ &= \lim_{h\to 0}\frac{c[f(x+h)-f(x)]}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= cf'(x)\end{aligned}$$

Therefore, $\frac{d}{dx}[cf(x)] = cf'(x)$.

Sum Rule

Sums of functions also behave under the same limit laws when we differentiate:

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\left[\frac{[f(x+h)-f(x)]}{h}+\frac{[g(x+h)-g(x)]}{h}\right]\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)+g'(x)\end{aligned}$$

So if f and g are differentiable at x,

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x).$$

The Sum Rule can be generalized for more than two functions to include n functions.

Note: Using the Sum Rule and the Constant Multiple Rule produces the Difference Rule:

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x).$$

Exercise

Using the differentiation rules we have discussed, calculate the derivatives of the following functions. Note which rule(s) you are using.

1. $y = x^5$
2. $y = 4x^3 - 2x^2$
3. $y = -1500$
4. $y = 3x^3 - 2x + 4$

Exponential Functions

Let $f(x) = b^x$, where $b > 0$, $b \neq 1$. To differentiate at 0, we write

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{b^x - b^0}{x} = \lim_{x \to 0} \frac{b^x - 1}{x}.$$

It is not obvious what this limit should be. However, consider the cases $b = 2$ and $b = 3$. By constructing a table of values, we can see that

$$\lim_{x \to 0} \frac{2^x - 1}{x} \approx 0.693 \quad \text{and} \quad \lim_{x \to 0} \frac{3^x - 1}{x} \approx 1.099.$$

So, $f'(0) < 1$ when $b = 2$ and $f'(0) > 1$ when $b = 3$. As it turns out, there is a particular number b, with $2 < b < 3$, whose graph has a tangent line with slope 1 at $x = 0$. In other words, such a number b has the property that

$$\lim_{x \to 0} \frac{b^x - 1}{x} = 1.$$

Question

What number is it?

Answer: This number is $e = 2.718281828459\ldots$ (known as the Euler number). The function $f(x) = e^x$ is called the **natural exponential function**.

Now, using $\lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$, we can find the formula for $\frac{d}{dx}(e^x)$:

$$\begin{aligned}
\frac{d}{dx}(e^x) &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} \\
&= e^x \left(\lim_{h\to 0} \frac{e^h - 1}{h} \right) \\
&= e^x \cdot 1 = e^x
\end{aligned}$$

Wed 17 Feb

- Expect Exam back on Thursday. Feedback on Friday. Scores $\rightarrow$ MLP?
- Instructions for when you get your exam back:
  - Look over your test, but don't write on it.
  - If you find discrepancies on points or grading, write your grievances on a separate sheet of paper.
  - Return that paper with your exam to your drill instructor by the end of drill.
  - Once you leave the room with your exam you lose this opportunity.
  - This is the only way you can get points back on the exam.

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Wed 17 Feb (cont.)

- MIDTERM in less than three weeks.
  - Tuesday 8 March 6-7:30p
  - If you have legitimate conflict, i.e., anything that is also scheduled in ISIS, I need to know now. If you are not sure if it conflicts with a course, please have that instructor contact me ASAP.
  - Morning Section: Walker rm 124
    Afternoon Section: Walker rm 218
- Later this month: Sub on Friday 26 Feb and Monday 29 Feb.

Exercise

(a) Find the slope of the line tangent to the curve $f(x) = x^3 - 4x - 4$ at the point $(2, -4)$.

(b) Where does this curve have a horizontal tangent?

Higher-Order Derivatives

If we can write the derivative of f as a function of x, then we can take *its* derivative, too. The derivative of the derivative is called the **second derivative** of f, and is denoted f''.

In general, we can differentiate f as often as needed. If we do it n times, the nth derivative of f is

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d}{dx}[f^{(n-1)}(x)].$$

3.3 Book Problems

9-48 (every 3rd problem), 51-53, 58-60

- For these problems, use only the rules we have derived so far.

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§3.4 The Product and Quotient Rules

Issue: Derivatives of products and quotients do NOT behave like they do for limits.

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As an example, consider $f(x) = x^2$ and $g(x) = x^3$. We can try to differentiate their product in two ways:

- $$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= \frac{d}{dx}\left(x^5\right)\\ &= 5x^4\end{aligned}$$
- $$\begin{aligned}f'(x)g'(x) &= (2x)(3x^2)\\ &= 6x^3\end{aligned}$$

Question

Which answer is the correct one?

Product Rule

If f and g are any two functions that are differentiable at x, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x).$$

In the example from the previous slide, we have

$$\begin{aligned}\frac{d}{dx}[x^2 \cdot x^3] &= \frac{d}{dx}(x^2) \cdot (x^3) + x^2 \cdot \frac{d}{dx}(x^3) \\ &= (2x) \cdot (x^3) + x^2 \cdot (3x^2) \\ &= 2x^4 + 3x^4 \\ &= 5x^4\end{aligned}$$

Derivation of the Product Rule

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h) + [-f(x)g(x+h) + f(x)g(x+h)] - f(x)g(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h}\right) + \left(\lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}\right)$$

Derivation of the Product Rule (cont.)

$$= \lim_{h\to 0}\left(g(x+h)\frac{f(x+h)-f(x)}{h}\right) + \left(\lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h}\right)$$

$$= g(x)f'(x) + f(x)g'(x)$$

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Exercise

Use the product rule to find the derivative of the function $(x^2 + 3x)(2x - 1)$.

A. $2(2x + 3)$

B. $6x^2 + 10x - 3$

C. $2x^3 + 5x^2 - 3x$

D. $2x(x + 3) + x(2x - 1)$

Derivation of Quotient Rule

Question

Let $q(x) = \frac{f(x)}{g(x)}$. What is $\frac{d}{dx}q(x)$?

Answer: We can write $f(x) = q(x)g(x)$ and then use the Product Rule:

$$f'(x) = q'(x)g(x) + g'(x)q(x)$$

and now solve for $q'(x)$:

$$q'(x) = \frac{f'(x) - q(x)g'(x)}{g(x)}.$$

Then, to get rid of $q(x)$, plug in $\frac{f(x)}{g(x)}$:

$$q'(x) = \frac{f'(x) - g'(x)\frac{f(x)}{g(x)}}{g(x)}$$

$$= \frac{g(x)\left(f'(x) - g'(x)\frac{f(x)}{g(x)}\right)}{g(x) \cdot g(x)}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

"LO-D-HI minus HI-D-LO over LO squared"

Quotient Rule

Just as with the product rule, the derivative of a quotient is not a quotient of derivatives, i.e.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}.$$

Here is the correct rule, the Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}.$$

Exercise

Use the Quotient Rule to find the derivative of

$$\frac{4x^3 + 2x - 3}{x + 1}.$$

Exercise

Find the slope of the tangent line to the curve

$$f(x) = \frac{2x - 3}{x + 1} \text{ at the point } (4, 1).$$

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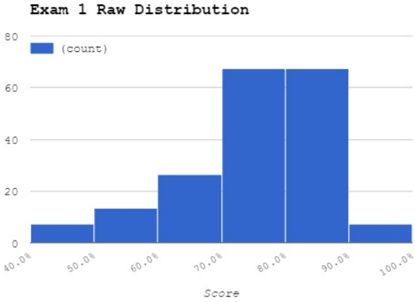
Mon 22 Feb

- Exam 1 Feedback

		Problem														
	Total	1	2	3(a)	(b)	(c)	(d)	4	5(a)	(b)	(c)	6(a)	(b)	(c)	7	8
out of	75	10	10	3	3	3	3	10	3	3	3	5	5	3	5	5
Median ->	48.0	8	7	2	0	1	2	8	2	3	1	5	1	1	4	3



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Mon 22 Feb (cont.)

- MIDTERM
  - Tuesday 8 March 6-7:30p
  - If you have legitimate conflict, i.e., anything that is also scheduled in ISIS, I need to know now. If you are not sure if it conflicts with a course, please have that instructor contact me ASAP.
  - Cumulative. Covers up to §3.9
  - Morning Section: Walker rm 124
    Afternoon Section: Walker rm 218
- Sub on Friday 26 Feb and Monday 29 Feb.
- Exam 2: Friday 4 March. Covers up to §3.8.

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The Quotient Rule also allows us to extend the Power Rule to negative numbers – if n is any integer, then

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}.$$

Question

How?

Exercise

If $f(x) = \frac{x(3-x)}{2x^2}$, find $f'(x)$.

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Derivative of e^{kx}

For any real number k,

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}.$$

Exercise

What is the derivative of x^2e^{3x}?

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Rates of Change

The derivative provides information about the instantaneous rate of change of the function being differentiated (compare to the limit of the slopes of the secant lines from §2.1).

For example, suppose that the population of a culture can be modeled by the function $p(t)$. We can find the instantaneous growth rate of the population at any time $t \geq 0$ by computing $p'(t)$ as well as the **steady-state population** (also called the **carrying capacity** of the population). The steady-state population equals

$$\lim_{t \to \infty} p(t).$$

3.4 Book Problems

9-49 (every 3rd problem), 57, 59, 63, 75-79 (odds)

Wed 24 Feb

- Exam 1: see the course webpage for the curve
- MIDTERM
  - Tuesday 8 March 6-7:30p
  - If you have legitimate conflict, i.e., anything that is also scheduled in ISIS, I need to know now. If you are not sure if it conflicts with a course, please have that instructor contact me ASAP.
  - Cumulative. Covers up to §3.9
  - Morning Section: Walker rm 124
    Afternoon Section: Walker rm 218

Wed 24 Feb (cont.)

- Sub on Friday 26 Feb and Monday 29 Feb.
- Possible sub on Wednesday 2 Mar.
- Exam 2: Friday 4 March. Covers up to §3.8.
- Quizzes: Only some of the quiz problems are graded now.

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5 15-19 February

6 22-26 February

- Monday 22 February
- Derivative of e^{kx}
- Rates of Change
- Book Problems
- Wed 24 February

§3.5 Derivatives of Trigonometric Functions

- Derivatives of Sine and Cosine Functions
- Trig Identities You Should Know
- Derivatives of Other Trig Functions
- Higher-Order Trig Derivatives
- Book Problems

§3.6 Derivatives as Rates of Change

- Position and Velocity
- Speed and Acceleration
- Growth Models
- Average and Marginal Cost
- Book Problems

7 29 Feb – 4 March

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§3.5 Derivatives of Trigonometric Functions

Trig functions are commonly used to model cyclic or periodic behavior in everyday settings. Therefore it is important to know how these functions change across time.

Fact: Derivative formulas for sine and cosine can be derived using the following limits:

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

(We will prove these limits in Chapter 4.)

Exercise

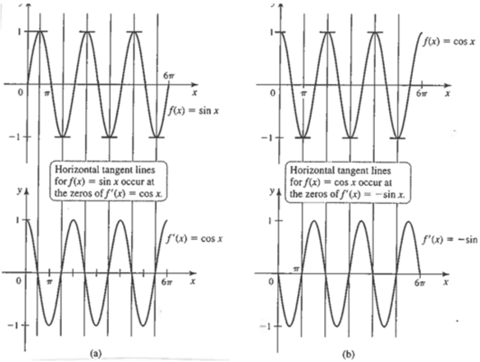
Evaluate $\lim_{x\to 0} \dfrac{\sin 9x}{x}$ and $\lim_{x\to 0} \dfrac{\sin 9x}{\sin 5x}$.

Derivatives of Sine and Cosine Functions

Using the previous limits and the definition of the derivative, we obtain

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

Examining the graphs of sine and cosine illustrate the relationship between the functions and their derivatives.



Trig Identities You Should Know

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = 1 - 2\sin^2 x$
- $\cos^2 x = \frac{1+\cos 2x}{2}$
- $\sin^2 x = \frac{1-\cos 2x}{2}$
- $\tan x = \frac{\sin x}{\cos x}$
- $\cot x = \frac{\cos x}{\sin x}$
- $\cot x = \frac{1}{\tan x}$
- $\sec x = \frac{1}{\cos x}$
- $\csc x = \frac{1}{\sin x}$

Derivatives of Other Trig functions

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cos x - (-\sin x)\sin x}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

So $\frac{d}{dx}(\tan x) = \sec^2 x$.

By using trig identities and the Quotient Rule, we obtain

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x$$
$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = -\csc^2 x$$

Exercise

Compute the derivative of the following functions:

$$f(x) = \frac{\tan x}{1 + \tan x} \qquad g(x) = \sin x \cos x$$

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

Exercise

Use the difference and product rules to find the derivative of the function $y = \cos x - x \sin x$.

A. $-\sin x + x \cos x$

B. $x \cos x$

C. $-2 \sin x - x \cos x$

D. $x \cos x - 2 \sin x$

Higher-Order Trig Derivatives

There is a cyclic relationship between the higher order derivatives of $\sin x$ and $\cos x$:

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned}$$

$$\begin{aligned} g(x) &= \cos x \\ g'(x) &= -\sin x \\ g''(x) &= -\cos x \\ g^{(3)}(x) &= \sin x \\ g^{(4)}(x) &= \cos x \end{aligned}$$

3.5 Book Problems

7-47 (odds), 57, 59, 61

5 15-19 February

6 22-26 February

- Monday 22 February
- Derivative of e^{kx}
- Rates of Change
- Book Problems
- Wed 24 February

§3.5 Derivatives of Trigonometric Functions

- Derivatives of Sine and Cosine Functions
- Trig Identities You Should Know
- Derivatives of Other Trig Functions
- Higher-Order Trig Derivatives
- Book Problems

§3.6 Derivatives as Rates of Change

- Position and Velocity
- Speed and Acceleration
- Growth Models
- Average and Marginal Cost
- Book Problems

7 29 Feb – 4 March

§3.6 Derivatives as Rates of Change

Question

Why do we need derivatives in real life?

We look at four areas where the derivative assists us with determining the rate of change in various contexts.

Position and Velocity

Suppose an object moves along a straight line and its location at time t is given by the position function $s = f(t)$. The **displacement** of the object between $t = a$ and $t = a + \Delta t$ is

$$\Delta s = f(a + \Delta t) - f(a).$$

Here Δt represents how much time has elapsed.

We now define average velocity as

$$\frac{\Delta s}{\Delta t} = \frac{f(a+\Delta t) - f(a)}{\Delta t}.$$

Recall that the limit of the average velocities as the time interval approaches 0 was the instantaneous velocity (which we denote here by v). Therefore, the instantaneous velocity at a is

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t} = f'(a).$$

Speed and Acceleration

In mathematics, speed and velocity are related but not the same – if the velocity of an object at any time t is given by $v(t)$, then the speed of the object at any time t is given by

$$|v(t)| = |f'(t)|.$$

By definition, acceleration (denoted by a) is the instantaneous rate of change of the velocity of an object at time t. Therefore,

$$a(t) = v'(t)$$

and since velocity was the derivative of the position function $s = f(t)$, then

$$a(t) = v'(t) = f''(t).$$

Summary: Given the position function $s = f(t)$, the velocity at time t is the first derivative, the speed at time t is the absolute value of the first derivative, and the acceleration at time t is the second derivative.

Question

Given the position function $s = f(t)$ of an object launched into the air, how would you know:

- The highest point the object reaches?
- How long it takes to hit the ground?
- The speed at which the object hits the ground?

Exercise

A rock is dropped off a bridge and its distance s (in feet) from the bridge after t seconds is $s(t) = 16t^2 + 4t$. At $t = 2$ what are, respectively, the velocity of the rock and the acceleration of the rock?

A. 64 ft/s; 16 ft/s^2
B. 68 ft/s; 32 ft/s^2
C. 64 ft/s; 32 ft/s^2
D. 68 ft/s; 16 ft/s^2

Growth Models

Suppose $p = f(t)$ is a function of the growth of some quantity of interest. The average growth rate of p between times $t = a$ and a later time $t = a + \Delta t$ is the change in p divided by the elapsed time Δt:

$$\frac{\Delta p}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

As Δt approaches 0, the average growth rate approaches the derivative $\frac{dp}{dt}$, which is the instantaneous growth rate (or just simply the growth rate). Therefore,

$$\frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}.$$

Exercise

The population of the state of Georgia (in thousands) from 1995 ($t = 0$) to 2005 ($t = 10$) is modeled by the polynomial

$$p(t) = -0.27t^2 + 101t + 7055.$$

(a) What was the average growth rate from 1995 to 2005?

(b) What was the growth rate for Georgia in 1997?

(c) What can you say about the population growth rate in Georgia between 1995 and 2005?

Average and Marginal Cost

Suppose a company produces a large amount of a particular quantity. Associated with manufacturing the quantity is a **cost function** $C(x)$ that gives the cost of manufacturing x items. This cost may include a **fixed cost** to get started as well as a **unit cost** (or **variable cost**) in producing one item.

If a company produces x items at a cost of $C(x)$, then the average cost is $\frac{C(x)}{x}$. This average cost indicates the cost of items already produced. Having produced x items, the cost of producing another Δx items is $C(x+\Delta x) - C(x)$. So the average cost of producing these extra Δx items is

$$\frac{\Delta C}{\Delta x} = \frac{C(x+\Delta x) - C(x)}{\Delta x}.$$

If we let Δx approach 0, we have

$$\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = C'(x)$$

which is called the **marginal cost**. The marginal cost is the approximate cost to produce one additional item after producing x items.

Note: In reality, we can't let Δx approach 0 because Δx represents whole numbers of items.

Exercise

If the cost of producing x items is given by

$$C(x) = -0.04x^2 + 100x + 800$$

for $0 \leq x \leq 1000$, find the average cost and marginal cost functions. Also, determine the average and marginal cost when $x = 500$.

3.6 Book Problems

9-19, 21-24, 30-33 (odds)

Wed 2 Mar

- Exam 2:
  - Friday 4 Mar. Covers up to §3.8.
  - Spring 2015 Practice Exam. Also look for quizzes on the old webpages for more problems.
  - For more problems study the evens in each of the sections covered.
  - Basic scientific calculator is allowed.

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Wed 2 Mar (cont.)

- Midterm:
  - Tuesday 8 March. Covers everything up to §3.9.
  - Morning Section: Walker room 124
    Afternoon Section: Walker room 218
    You must take the test with your officially scheduled section.
  - Stay tuned for conflict resolutions. If you haven't emailed me already regarding a conflict, do it NOW.
  - Stay tuned for a study guide.
  - Basic scientific calculator is allowed....? Stay tuned.

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Wed 2 Mar (cont.)

- Quiz 6 next Thurs. Only some of the quiz problems are graded now. You are always welcome to my office for feeback on your work.

§3.7 The Chain Rule

The rules up to now have not allowed us to differentiate composite functions

$$f \circ g(x) = f(g(x)).$$

Example

If $f(x) = x^7$ and $g(x) = 2x - 3$, then $f(g(x)) = (2x - 3)^7$. To differentiate we could mulitply the polynomial out... but in general we should use a much more efficient strategy to emply to composition functions.

Example

Suppose that Yvonne (y) can run twice as fast as Uma (u). Then write $\frac{dy}{du} = 2$.

Suppose that Uma can run four times as fast as Xavier (x). So $\frac{du}{dx} = 4$.

How much faster can Yvonne run than Xavier? In this case, we would take both our rates and multiply them together:

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 4 = 8.$$

Version 1 of the Chain Rule

If g is differentiable at x, and $y = f(u)$ is differentiable at $u = g(x)$, then the composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Guidelines for Using the Chain Rule

Assume the differentiable function $y = f(g(x))$ is given.

1. Identify the outer function f, the inner function g, and let $u = g(x)$.
2. Replace $g(x)$ by u to express y in terms of u:

$$y = f(g(x)) \implies y = f(u)$$

3. Calculate the product $\frac{dy}{du} \cdot \frac{du}{dx}$
4. Replace u by $g(x)$ in $\frac{dy}{du}$ to obtain $\frac{dy}{dx}$.

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Example

Use Version 1 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (5x^2 + 11x)^{20}$.

- inner function: $u = 5x^2 + 11x$
- outer function: $y = u^{20}$

We have $y = f(g(x)) = (5x^2 + 11x)^{20}$. Differentiate:

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} &= 20u^{19} \cdot (10x + 11) \\ &= 20(5x^2 + 11x)^{19} \cdot (10x + 11)\end{aligned}$$

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Exercise

Use the first version of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \left(\frac{3x}{4x+2}\right)^5.$$

Exercise

Use the first version of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \cos(5x+1).$$

A. $y' = -\cos(5x+1) \cdot \sin(5x+1)$
B. $y' = -5\sin(5x+1)$
C. $y' = 5\cos(5x+1) - \sin(5x+1)$
D. $y' = -\sin(5x+1)$

Version 2 of the Chain Rule

Notice if $y = f(u)$ and $u = g(x)$, then $y = f(u) = f(g(x))$, so we can also write:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= f'(u) \cdot g'(x)$$

$$= f'(g(x)) \cdot g'(x).$$

Example

Use Version 2 of the Chain Rule to calculate $\frac{dy}{dx}$ for $y = (7x^4 + 2x + 5)^9$.

- inner function: $g(x) = 7x^4 + 2x + 5$
- outer function: $f(u) = u^9$

Then

$$f'(u) = 9u^8 \implies f'(g(x)) = 9(7x^4 + 2x + 5)^8$$
$$g'(x) = 28x^3 + 2.$$

Putting it together,

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = 9(7x^4 + 2x + 5)^8 \cdot (28x^3 + 2)$$

Exercise

Use Version 2 of the Chain Rule to calculate $\frac{dy}{dx}$ for

$$y = \tan\left(5x^5 - 7x^3 + 2x\right).$$

Chain Rule for Powers

If g is differentiable for all x in the domain and n is an integer, then

$$\frac{d}{dx}\Big[(g(x))^n\Big] = n(g(x))^{n-1} \cdot g'(x).$$

Chain Rule for Powers (cont.)

Example

$\frac{d}{dx}\left[(1-e^x)^4\right] = ?$

Answer:

$$\frac{d}{dx}\left[(1-e^x)^4\right] = 4(1-e^x)^3 \cdot (-e^x)$$
$$= -4e^x(1-e^x)^3$$

Composition of 3 or More Functions

Example

Compute $\frac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right]$.

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Composition of 3 or More Functions (cont.)

Answer:

$$\begin{aligned}\frac{d}{dx}\left[\sqrt{(3x-4)^2+3x}\right] &= \frac{1}{2}\left((3x-4)^2+3x\right)^{-\frac{1}{2}}\cdot\frac{d}{dx}\left[(3x-4)^2+3x\right]\\ &= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}}\cdot\left[2(3x-4)\frac{d}{dx}(3x-4)+3\right]\\ &= \frac{1}{2\sqrt{\left((3x-4)^2+3x\right)}}\cdot\left[2(3x-4)\cdot 3+3\right]\\ &= \frac{18x-21}{2\sqrt{\left((3x-4)^2+3x\right)}}\end{aligned}$$

3.7 Book Problems

7-33 (odds), 38, 45-67 (odds)

5 15-19 February

6 22-26 February

7 29 Feb – 4 March

- Wednesday 2 March

§3.7 The Chain Rule

- Version 1 of the Chain Rule
- Guidelines for Using the Chain Rule
- Version 2 of the Chain Rule
- Chain Rule for Powers
- Composition of 3 or More Functions
- Book Problems

§3.8 Implicit Differentiation

- Higher Order Derivatives
- Power Rule for Rational Exponents
- Book Problems

Exam #2 Review

- Running Out of Time on the Exam Plus other Study Tips
- Other Study Tips

§3.8 Implicit Differentiation

Up to now, we have calculated derivatives of functions of the form $y = f(x)$, where y is defined **explicitly** in terms of x. In this section, we examine relationships between variables that are **implicit** in nature, meaning that y either is not defined explicitly in terms of x or cannot be easily manipulated to solve for y in terms of x.

The goal of **implicit differentiation** is to find a single expression for the derivative directly from an equation of the form $F(x, y) = 0$ without first solving for y.

Example

Calculate $\frac{dy}{dx}$ directly from the equation for the circle

$$x^2 + y^2 = 9.$$

Solution: To remind ourselves that x is our independent variable and that we are differentiating with respect to x, we can replace y with $y(x)$:

$$x^2 + (y(x))^2 = 9.$$

Now differentiate each term with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}((y(x))^2) = \frac{d}{dx}(9).$$

By the Chain Rule, $\frac{d}{dx}((y(x))^2) = 2y(x)y'(x)$ (Version 2), or $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ (Version 1). So

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \implies \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= -\frac{x}{y}. \end{aligned}$$

The derivative is a function of x and y, meaning we can write it in the form

$$F(x,y) = -\frac{x}{y}.$$

To find slopes of tangent lines at various points along the circle we just plug in the coordinates. For example, the slope of the tangent line at (0,3) is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,3)} = -\frac{0}{3} = 0.$$

The slope of the tangent line at $(1, 2\sqrt{2})$ is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2\sqrt{2})} = -\frac{1}{2\sqrt{2}}.$$

The point is that, in some cases it is difficult to solve an implicit equation in terms of y and then differentiate with respect to x. In other cases, although it may be easier to solve for y in terms of x, you may need two or more functions to do so, which means two or more derivatives must be calculated (e.g., circles).

The goal of implicit differentiation is to find one single expression for the derivative directly given $F(x, y) = 0$ (i.e., some equation with xs and ys in it), without solving first for y.

Question

The following functions are implicitly defined:

- $x + y^3 - xy = 4$
- $\cos(x - y) + \sin y = \sqrt{2}$

For each of these functions, how would you find $\frac{dy}{dx}$?

Exercise

Find $\frac{dy}{dx}$ for $xy + y^3 = 1$.

Exercise

Find an equation of the line tangent to the curve $x^4 - x^2y + y^4 = 1$ at the point $(-1, 1)$.

Higher Order Derivatives

Example

Find $\frac{d^2y}{dx^2}$ if $xy + y^3 = 1$.

Exercise

If $\sin x = \sin y$, then $\frac{dy}{dx} = ?$ and $\frac{d^2y}{dx^2} = ?$

A. $\frac{\cos y}{\cos x}$; $\frac{\tan y \cos^2 x - \sin x \cos y}{\cos^2 x}$

B. $\frac{\cos x}{\cos y}$; $\frac{\tan y \cos^2 x - \sin x \cos y}{\cos^2 y}$

C. $\frac{\cos x}{\cos y}$; $\frac{\cos y(\sin x - \sin y)}{\cos^2 y}$

D. $\frac{\cos y}{\cos x}$; $\frac{\cos y(\sin x - \sin y)}{\cos^2 x}$

Power Rule for Rational Exponents

Implicit differentiation also allows us to extend the power rule to rational exponents: Assume p and q are integers with $q \neq 0$. Then

$$\frac{d}{dx}(x^{\frac{p}{q}}) = \frac{p}{q}x^{\frac{p}{q}-1}$$

(provided $x \geq 0$ when q is even and $\frac{p}{q}$ is in lowest terms).

Exercise

Prove it.

3.8 Book Problems

5-25 (odds), 31-49 (odds)

5 15-19 February

6 22-26 February

7 29 Feb – 4 March
- Wednesday 2 March

§3.7 The Chain Rule
- Version 1 of the Chain Rule
- Guidelines for Using the Chain Rule
- Version 2 of the Chain Rule
- Chain Rule for Powers
- Composition of 3 or More Functions
- Book Problems

§3.8 Implicit Differentiation
- Higher Order Derivatives
- Power Rule for Rational Exponents
- Book Problems

Exam #2 Review
- Running Out of Time on the Exam Plus other Study Tips
- Other Study Tips

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Exam #2 Review

- §3.2 Working with Derivatives
  - Be able to use the graph of a function to sketch the graph of its derivative, without computing derivatives
  - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point
  - Be able to determine where a function is not differentiable

Exam #2 Review (cont.)

- §3.3 Rules for Differentiation
  - Be able to use the various rules for differentiation (e.g., constant rule, power rule, constant multiple rule, sum and difference rule) to calculate the derivative of a function.
  - Know the derivative of e^x.
  - Be able to find slopes and/or equations of tangent lines.
  - Be able to calculate higher-order derivatives of functions.

Exam #2 Review (cont.)

Exercise

Given that $y = 3x + 2$ is tangent to $f(x)$ at $x = 1$ and that $y = -5x + 6$ is tangent to $g(x)$ at $x = 1$, write the equation of the tangent line to $h(x) = f(x)g(x)$ at $x = 1$.

Exam #2 Review (cont.)

- §3.4 The Product and Quotient Rules
  - Be able to use the product and/or quotient rules to calculate the derivative of a given function.
  - Be able to use the product and/or quotient rules to find tangent lines and/or slopes at a given point.
  - Know the derivative of e^{kx}.
  - Be able to combine derivative rules to calculate the derivative of a function.

Note: Functions are not always given by a formula. When faced with a problem where you don't know where to start, go through the rules first.

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Exam #2 Review (cont.)

Exercise

Suppose you have the following information about the functions f and g:

$$f(1) = 6 \quad f'(1) = 2 \quad g(1) = 2 \quad g'(1) = 3$$

- Let $F = 2f + 3g$. What is $F(1)$? What is $F'(1)$?
- Let $G = fg$. What is $G(1)$? What is $G'(1)$?

Exam #2 Review (cont.)

- §3.5 Derivatives of Trigonometric Functions
  - Know the two special trigonometric limits

    $$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

    and be able to use them to solve other similar limits.
  - Know the derivatives of $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$, and be able to use the quotient rule to derive the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$.
  - Be able to calculate derivatives (including higher order) involving trig functions using the rules for differentiation.

Exam #2 Review (cont.)

Exercise

Calculate the derivative of the following functions:

- $f(x) = (1 + \sec x)\sin^3 x$
- $g(x) = \dfrac{\sin x + \cot x}{\cos x}$

Exercise

Evaluate $\displaystyle\lim_{x \to -3} \frac{\sin(x+3)}{x^2 + 8x + 15}$.

Exam #2 Review (cont.)

- §3.6 Derivatives as Rates of Change
  - Be able to use the derivative to answer questions about rates of change involving:
    - Position and velocity
    - Speed and acceleration
    - Growth rates
    - Business applications

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Exam #2 Review (cont.)

- Be able to use a position function to answer questions involving velocity, speed, acceleration, height/distance at a particular time t, maximum height, and time at which a given height/distance is achieved.
- Be able to use growth models to answer questions involving growth rate and average growth rate, and cost functions to answer questions involving average and marginal costs.

Exam #2 Review (cont.)

- §3.7 The Chain Rule
  - Be able to use both versions of the Chain Rule to find the derivative of a composition function.
  - Be able to use the Chain Rule more than once in a calculation involving more than two composed functions.
  - Know and be able to use the Chain Rule for Powers:

$$\frac{d}{dx}\left(f(x)\right)^n = n\left(f(x)\right)^{n-1} f'(x)$$

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Exam #2 Review (cont.)

Exercise

Suppose $f(9) = 10$ and $g(x) = f(x^2)$. What is $g'(3)$?

Exam #2 Review (cont.)

- §3.7 Implicit Differentiation
  - Be able to use implicit differentiation to calculate $\frac{dy}{dx}$.
  - Be able to use the derivative found from implicit differentiation to find the slope at a given point and/or a line tangent to the curve at the given point.
  - Be able to calculate higher-order derivatives of implicitly defined functions.
  - Be able to calculate $\frac{dy}{dx}$ when working with functions containing rational functions.

Exam #2 Review (cont.)

Exercise

Use implicit differentiation to calculate $\frac{dz}{dw}$ for

$$e^{2w} = \sin(wz)$$

Exercise

If $\sin x = \sin y$, then

- $\frac{dy}{dx} = ?$
- $\frac{d^2y}{dx^2} = ?$

Running Out of Time on the Exam Plus other Study Tips

- Do practice problems completely, from beginning to end (as if it were a quiz). You might think you understand something but when it's time to write down the details things are not so clear.
- Find a buddy who understands concepts a little better than you and work on problems for 2-3 hours. Then find a buddy who is struggling and work with them 2-3 hours.
- Don't count on cookie cutter problems. If you are doing a practice problem where you've memorized all the steps, make sure you understand why each step is needed. The exam problems may have a small variation from homeworks and quizzes. If you're not prepared, it'll come as a "twist" on the exam...

Running Out of Time on the Exam Plus other Study Tips (cont.)

- If you encounter an unfamiliar type of problem on the exam, relax, because it's most likely not a trick! The solutions will always rely on the information from the required reading/assignments. Take your time and do each baby step carefully.
- During the exam, do the problems you are most confident with first!
- During the exam, budget your time. Count the problems and divide by 50 minutes. The easier questions will take less time so doing them first leaves extra time for the harder ones. When studying, aim for 10 problems per hour (i.e., 6 minutes per problem).

Running Out of Time on the Exam Plus other Study Tips (cont.)

- Always make sure you answer the question. This is also a good strategy if you're not sure how to start a problem, figure out what the question wants first.
- The exam is not a race. If you finish early take advantage of the time to check your work. You don't want to leave feeling smug about how quickly you finished only to find out next week you lost a letter grade's worth of points from silly mistakes.

Other Study Tips

- Brush up on algebra, especially radicals, logs, common denominators, etc. Many times knowing the right algebra will simplify the problem!
- When in doubt, show steps.
- You will be punished for wrong notation. The slides for §3.1 show different notations for the derivative. Make sure whichever one you use in your work, that you are using it correctly.
- Read the question!
- Do the book problems.
- Look at the pictures in the book and the interactive applets on MLP.

Part 3. Applications and Story Problems

8 7-11 March

- **Monday 7 March**

§3.9 Derivatives of Logarithmic and Exponential Functions

- Derivative of $y = \ln x$
- Derivative of $y = \ln |x|$
- Derivative of $y = b^x$
- Story Problem Example
- Derivatives of General Logarithmic Functions
- Neat Trick: Logarithmic Differentiation
- Book Problems

Midterm Review

- Running Out of Time on the Exam Plus other Study Tips
- Other Study Tips
- **Friday 11 March**

§3.10 Derivatives of Inverse Trigonometric Functions

- Derivative of Inverse Sine
- Derivative of Inverse Tangent
- Derivative of Inverse Secant
- All Other Inverse Trig Derivatives
- Derivatives of Inverse Functions in General
- Book Problems

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§3.11 Related Rates

Part 3. Applications and Story Problems (cont.)

9. 14-18 March
    - **Monday 14 March**
    - Steps for Solving Related Rates Problems
    - Book Problems
    - **Wednesday 16 March**

§4.1 Maxima and Minima

- Extreme Value Theorem
- Local Maxima and Minima
- Critical Points
- Local Extreme Point Theorem
- Locating Absolute Min and Max
- Book Problems
- **Friday 18 March**

§4.2 What Derivatives Tell Us

- How is it related to the derivative?
- First Derivative Test
- Absolute extremes on any interval
- Derivative of the derivative tells us:
- Test for Concavity
- Second Derivative Test
- Book Problems

10. 28 Mar – 1 April
    - **Monday 28 March**

Part 3. Applications and Story Problems (cont.)

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

Mon 7 Mar

- Exam 2: Solutions posted as soon as make-ups are in. But we will go through them today. You'll get your test back in drill tomorrow and the curve will be posted.
- Midterm:
  - Covers everything up to §3.9. All the slides are up. We will work fast through them today, but solutions to the exercises in the slides will be posted.
  - Morning Section: Walker room 124
    Afternoon Section: Walker room 218
    You must take the test with your officially scheduled section.

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Mon 7 Mar (cont.)

- If you have questions about your exam conflicts, contact me NOW.
- Study guide is in MLP.
- Basic scientific calculator is allowed...? Yes.
- Sit in every other seat.
- 15 Questions, 10 points each.
- Don't expect a curve. :(

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§3.9 Derivatives of Logarithmic and Exponential Functions

The natural exponential function $f(x) = e^x$ has an inverse function, namely $f^{-1}(x) = \ln x$. This relationship has the following properties:

1. $e^{\ln x} = x$ for $x > 0$ and $\ln(e^x) = x$ for all x.
2. $y = \ln x \iff x = e^y$
3. For real numbers x and $b > 0$,

$$b^x = e^{\ln(b^x)} = e^{x \ln b}.$$

Derivative of $y = \ln x$

Using 2. from the last slide, plus implicit differentiation, we can find $\frac{d}{dx}(\ln x)$. Write $y = \ln x$. We wish to find $\frac{dy}{dx}$. From 2.,

$$\frac{d}{dx}(x = e^y) \Rightarrow \frac{d}{dx}x = \frac{d}{dx}(e^y)$$

$$1 = e^y \left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

So $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

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Derivative of $y = \ln|x|$

Recall, we can only take "$\ln$" of a positive number. However:

- For $x > 0$, $\ln|x| = \ln x$, so
$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$
- For $x < 0$, $\ln|x| = \ln(-x)$, so
$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}.$$

In other words, the absolute values do not change the derivative of natural log.

Exercise

Find the derivative of each of the following functions:

- $f(x) = \ln(15x)$
- $g(x) = x \ln x$
- $h(x) = \ln(\sin x)$

Derivative of $y = b^x$

What about other logs? Say $b > 0$. Since $b^x = e^{\ln b^x} = e^{x \ln b}$ (by 3. on the earlier slide),

$$\begin{aligned} \frac{d}{dx}(b^x) &= \frac{d}{dx}(e^{x \ln b}) \\ &= e^{x \ln b} \cdot \ln b \\ &= b^x \ln b. \end{aligned}$$

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Exercise

Find the derivative of each of the following functions:

- $f(x) = 14^x$
- $g(x) = 45(3^{2x})$

Exercise

Determine the slope of the tangent line to the graph $f(x) = 4^x$ at $x = 0$.

Story Problem Example

Example

The energy (in Joules) released by an earthquake of magnitude M is given by the equation

$$E = 25000 \cdot 10^{1.5M}.$$

(a) How much energy is released in a magnitude 3.0 earthquake?

(b) What size earthquake releases 8 million Joules of energy?

(c) What is $\frac{dE}{dM}$ and what does it tell you?

Derivatives of General Logarithmic Functions

The relationship $y = \ln x \iff x = e^y$ applies to logarithms of other bases:

$$y = \log_b x \quad \iff \quad x = b^y.$$

Now taking $\frac{d}{dx}(x = b^y)$ we obtain

$$1 = b^y \ln b \left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln b}$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

Example

The derivative of $f(x) = \log_2(10x)$ is

A. $\frac{1}{10x}$

B. $\frac{1}{x \ln 2}$

C. $\frac{1}{x}$

D. $\frac{10}{x \ln 2}$

Neat Trick: Logarithmic Differentiation

Example

Compute the derivative of $f(x) = \frac{x^2(x-1)^3}{(3+5x)^4}$.

Solution: We can use logarithmic differentiation – first take the natural log of both sides and then use properties of logarithms.

$$\begin{aligned}\ln(f(x)) &= \ln\left(\frac{x^2(x-1)^3}{(3+5x)^4}\right)\\ &= \ln x^2 + \ln{(x-1)^3} - \ln{(3+5x)^4}\\ &= 2\ln x + 3\ln(x-1) - 4\ln(3+5x)\end{aligned}$$

Now we take $\frac{d}{dx}$ on both sides:

$$\frac{1}{f(x)}\left(\frac{df}{dx}\right) = 2\left(\frac{1}{x}\right) + 3\left(\frac{1}{x-1}\right) - 4\left(\frac{1}{3+5x}\right) (5)$$

$$\frac{f'(x)}{f(x)} = \frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}$$

Finally, solve for $f'(x)$:

$$f'(x) = f(x)\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

$$= \frac{x^2(x-1)^3}{(3+5x)^4}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{20}{3+5x}\right]$$

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Exercise

Use logarithmic differentiation to calculate the derivative of

$$f(x) = \frac{(x+1)^{\frac{3}{2}}(x-4)^{\frac{5}{2}}}{(5x+3)^{\frac{2}{3}}}.$$

3.9 Book Problems

9-29 (odds), 55-67 (odds)

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Midterm Review

- §2.1-2.2
  - **Material may not be explicitly tested, but the topics here are foundational to later sections.**
- §2.3 Techniques for Computing Limits
  - **Be able to do questions similar to 1-48.**
  - Know and be able to compute limits using analytical methods (e.g., limit laws, additional techniques).
  - Be able to evaluate one-sided and two-sided limits of functions.
  - Know the Squeeze Theorem and be able to use this theorem to determine limits.

Midterm Review (cont.)

Exercise (problems from past midterm)

Evaluate the following limits:

- $\displaystyle\lim_{x\to 3}\frac{x^2-x-6}{x^2-9}$
- $\displaystyle\lim_{\theta\to 0}\frac{\sec\theta\tan\theta}{\theta}$

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Midterm Review (cont.)

- §2.4 Infinite Limits
  - **Be able to do questions similar to 17-30.**
  - Be able to use a graph, a table, or analytical methods to determine infinite limits.
  - Be able to use analytical methods to evaluate one-sided limits.
  - Know the definition of a vertical asymptote and be able to determine whether a function has vertical asymptotes.

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Midterm Review (cont.)

- §2.5 Limits at Infinity
  - **Be able to do questions similar to 9-30 and 38-46.**
  - Be able to find limits at infinity and horizontal asymptotes.
  - Know how to compute the limits at infinity of rational functions and algebraic functions.
  - Be able to list horizontal and/or vertical asymptotes of a function.

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Midterm Review (cont.)

Exercise

Determine the horizontal asymptote(s) for the function

$$f(x) = \frac{10x^3 - 3x^2 + 8}{\sqrt{25x^6 + x^4 + 2}}$$

A. $y = 2$

B. $y = 0$

C. $y = -2$

D. $y = \pm 2$

Midterm Review (cont.)

- §2.6 Continuity
  - **Be able to do questions similar to 9-44.**
  - Know the definition of continuity and be able to apply the continuity checklist.
  - Be able to determine the continuity of a function (including those with roots) on an interval.
  - Be able to apply the Intermediate Value Theorem to a function.

Midterm Review (cont.)

Exercise (problem from past midterm)

Determine the value of k so the function is continuous on $0 \leq x \leq 2$.

$$f(x) = \begin{cases} x^2 + k & 0 \leq x \leq 1 \\ -2kx + 4 & 1 < x \leq 2 \end{cases}$$

Midterm Review (cont.)

- §3.1 Introducing the Derivative
  - **Be able to do questions similar to 11-32.**
  - Know the definition of a derivative and be able to use this definition to calculate the derivative of a given function.
  - Be able to determine the equation of a line tangent to the graph of a function at a given point.
  - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point.

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Midterm Review (cont.)

- §3.2 Working with Derivatives
  - Be able to use the graph of a function to sketch the graph of its derivative, without computing derivatives
  - Know the 3 conditions for when a function is not differentiable at a point, and why these three conditions make a function not differentiable at the given point
  - Be able to determine where a function is not differentiable

Midterm Review (cont.)

- §3.3 Rules for Differentiation
  - **Be able to do questions similar to 7-41.**
  - Be able to use the various rules for differentiation (e.g., constant rule, power rule, constant multiple rule, sum and difference rule) to calculate the derivative of a function.
  - Know the derivative of e^x.
  - Be able to find slopes and/or equations of tangent lines.
  - Be able to calculate higher-order derivatives of functions.

Midterm Review (cont.)

Exercise

Given that $y = 3x + 2$ is tangent to $f(x)$ at $x = 1$ and that $y = -5x + 6$ is tangent to $g(x)$ at $x = 1$, write the equation of the tangent line to $h(x) = f(x)g(x)$ at $x = 1$.

Midterm Review (cont.)

- §3.4 The Product and Quotient Rules
  - **Be able to do questions similar to 7-42 and 47-52.**
  - Be able to use the product and/or quotient rules to calculate the derivative of a given function.
  - Be able to use the product and/or quotient rules to find tangent lines and/or slopes at a given point.
  - Know the derivative of e^{kx}.
  - Be able to combine derivative rules to calculate the derivative of a function.

Note: Functions are not always given by a formula. When faced with a problem where you don't know where to start, go through the rules first.

Midterm Review (cont.)

Exercise

Suppose you have the following information about the functions f and g:

$$f(1) = 6 \quad f'(1) = 2 \quad g(1) = 2 \quad g'(1) = 3$$

- Let $F = 2f + 3g$. What is $F(1)$? What is $F'(1)$?
- Let $G = fg$. What is $G(1)$? What is $G'(1)$?

Midterm Review (cont.)

- §3.5 Derivatives of Trigonometric Functions
  - **Be able to do questions similar to 1-55.**
  - Know the two special trigonometric limits

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{\cos x-1}{x}=0$$

    and be able to use them to solve other similar limits.
  - Know the derivatives of $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$, and be able to use the quotient rule to derive the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$.
  - Be able to calculate derivatives (including higher order) involving trig functions using the rules for differentiation.

Midterm Review (cont.)

Exercise

Calculate the derivative of the following functions:

- $f(x) = (1 + \sec x)\sin^3 x$
- $g(x) = \dfrac{\sin x + \cot x}{\cos x}$

Exercise

Evaluate $\displaystyle\lim_{x \to -3} \frac{\sin(x+3)}{x^2 + 8x + 15}$.

Midterm Review (cont.)

- §3.6 Derivatives as Rates of Change
  - **Be able to do questions similar to 11-18.**
  - Be able to use the derivative to answer questions about rates of change involving:
    - Position and velocity
    - Speed and acceleration
    - Growth rates
    - Business applications

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Midterm Review (cont.)

- Be able to use a position function to answer questions involving velocity, speed, acceleration, height/distance at a particular time t, maximum height, and time at which a given height/distance is achieved.
- Be able to use growth models to answer questions involving growth rate and average growth rate, and cost functions to answer questions involving average and marginal costs.

Midterm Review (cont.)

- §3.7 The Chain Rule
  - **Be able to do questions similar to 7-43.**
  - Be able to use both versions of the Chain Rule to find the derivative of a composition function.
  - Be able to use the Chain Rule more than once in a calculation involving more than two composed functions.
  - Know and be able to use the Chain Rule for Powers:

$$\frac{d}{dx}\left(f(x)\right)^n = n\left(f(x)\right)^{n-1} f'(x)$$

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Midterm Review (cont.)

Exercise

Suppose $f'(9) = 10$ and $g(x) = f(x^2)$. What is $g'(3)$?

Midterm Review (cont.)

- §3.8 Implicit Differentiation
  - **Be able to do questions similar to 5-26 and 33-46.**
  - Be able to use implicit differentiation to calculate $\frac{dy}{dx}$.
  - Be able to use the derivative found from implicit differentiation to find the slope at a given point and/or a line tangent to the curve at the given point.
  - Be able to calculate higher-order derivatives of implicitly defined functions.
  - Be able to calculate $\frac{dy}{dx}$ when working with functions containing rational functions.

Midterm Review (cont.)

Exercise

Use implicit differentiation to calculate $\frac{dz}{dw}$ for

$$e^{2w} = \sin(wz)$$

Exercise

If $\sin x = \sin y$, then

- $\frac{dy}{dx} = ?$
- $\frac{d^2y}{dx^2} = ?$

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Midterm Review (cont.)

- §3.9 Derivatives of Logarithmic and Exponential Functions
  - Be able to compute derivatives involving $\ln x$ and $\log_b x$
  - Be able to compute derivatives of exponential functions of the form b^x
  - Be able to use logarithmic differentiation to determine $f'(x)$

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Running Out of Time on the Exam Plus other Study Tips

- Do practice problems completely, from beginning to end (as if it were a quiz). You might think you understand something but when it's time to write down the details things are not so clear.
- Find a buddy who understands concepts a little better than you and work on problems for 2-3 hours. Then find a buddy who is struggling and work with them 2-3 hours.
- Don't count on cookie cutter problems. If you are doing a practice problem where you've memorized all the steps, make sure you understand why each step is needed. The exam problems may have a small variation from homeworks and quizzes. If you're not prepared, it'll come as a "twist" on the exam...

Running Out of Time on the Exam Plus other Study Tips (cont.)

- If you encounter an unfamiliar type of problem on the exam, relax, because it's most likely not a trick! The solutions will always rely on the information from the required reading/assignments. Take your time and do each baby step carefully.
- During the exam, do the problems you are most confident with first!
- During the exam, budget your time. Count the problems and divide by 50 minutes. The easier questions will take less time so doing them first leaves extra time for the harder ones. When studying, aim for 10 problems per hour (i.e., 6 minutes per problem).

Running Out of Time on the Exam Plus other Study Tips (cont.)

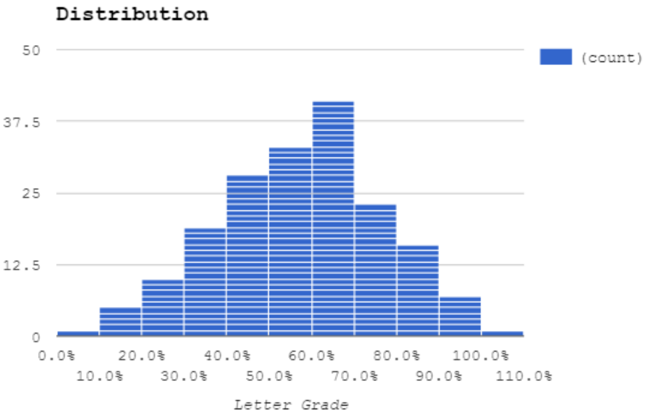
- Always make sure you answer the question. This is also a good strategy if you're not sure how to start a problem, figure out what the question wants first.
- The exam is not a race. If you finish early take advantage of the time to check your work. You don't want to leave feeling smug about how quickly you finished only to find out next week you lost a letter grade's worth of points from silly mistakes.

Other Study Tips

- Brush up on algebra, especially radicals, logs, common denominators, etc. Many times knowing the right algebra will simplify the problem!
- When in doubt, show steps.
- You will be punished for wrong notation. The slides for §3.1 show different notations for the derivative. Make sure whichever one you use in your work, that you are using it correctly.
- Read the question!
- Do the book problems.
- Look at the pictures in the book and the interactive applets on MLP.

Fri 11 Mar

- Exam 2: Curve, etc. is posted.



Fri 11 Mar (cont.)

- Midterm: expect it back next week in drill. Don't expect a curve. :(
- "Fast Track Calculus": Dr. Kathleen Morris will be teaching a second 8 weeks Calculus One class. "If you have a student who is maybe doing poorly because of illness or a tragic event in their life during the beginning of the semester, this might be an opportunity for a new start for them." The class requires departmental consent so the student will need to contact Kathleen to get permission to enroll.

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§3.10 Derivatives of Inverse Trigonometric Functions

Recall: If $y = f(x)$, then $f^{-1}(x)$ is the value of y such that $x = f(y)$.

Example

If $f(x) = 3x + 2$, then what is $f^{-1}(x)$?

NOTE: $f^{-1}(x) \neq f(x)^{-1} \left(= \frac{1}{f(x)}\right)$

To avoid this confusion, we use $\arcsin x$, $\arccos x$ $\arctan x, \ldots$ to denote inverse trig functions.

Derivative of Inverse Sine

Trig functions are functions, too. Just like with "f", there has to be something to "plug in". It makes no sense to just say $\sin$, without having $\sin($*something*$)$.

$$y = \sin^{-1} x \Longleftrightarrow x = \sin y$$

The derivative of $y = \sin^{-1} x$ can be found using implicit differentiation:

$$\begin{aligned} x &= \sin y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\sin y) \\ 1 &= (\cos y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\cos y} \end{aligned}$$

We still need to replace $\cos y$ with an expression in terms of x. We use the trig identity $\sin^2 y + \cos^2 y = 1$ (careful with notation: in this case we mean $(\sin y)^2 + (\cos y)^2 = 1$). Then

$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}.$$

The range of $y = \sin^{-1} x$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. In this range, cosine is never negative, so we can just take the positive portion of the square root. Therefore,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}} \implies \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Exercise

Compute the following:

1. $\frac{d}{dx}\left(\sin^{-1}(4x^2-3)\right)$
2. $\frac{d}{dx}\left(\cos(\sin^{-1}x)\right)$

Derivative of Inverse Tangent

Similarly to inverse sine, we can let $y = \tan^{-1} x$ and use implicit differentiation:

$$\begin{aligned} x &= \tan y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\tan y) \\ 1 &= (\sec^2 y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \end{aligned}$$

Use the trig identity $\sec^2 y - \tan^2 y = 1$ to replace $\sec^2 y$ with $1 + x^2$:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

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Derivative of Inverse Secant

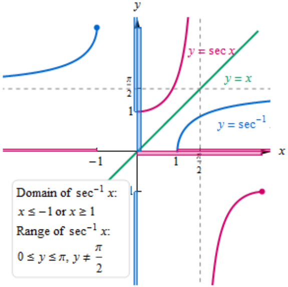
Again, use the same method as with inverse sine:

$$\begin{aligned} y &= \sec^{-1} x \\ x &= \sec y \\ \frac{d}{dx}(x) &= \frac{d}{dx}(\sec y) \\ 1 &= \sec y \tan y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y} \end{aligned}$$

Use the trig identity $\sec^2 y - \tan^2 y = 1$ again to get

$$\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}.$$

This time, the $\pm$ matters:



- If $x \geq 1$, then $0 \leq y < \frac{\pi}{2}$ and so $\tan y > 0$.
- If $x \leq -1$, then $\frac{\pi}{2} < y \leq \pi$ and so $\tan y < 0$.

Therefore,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}.$$

Using other trig identities (which you do not need to prove)

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2} \quad \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2} \quad \csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

we can get the rest of the inverse trig derivatives.

All Other Inverse Trig Derivatives

To summarize:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$
$$(-1 < x < 1)$$
$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$
$$(-\infty < x < \infty)$$
$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$
$$(|x| > 1)$$

Example

Compute the derivatives of $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$ and $g(x) = \sin\left(\sec^{-1}(2x)\right)$.

Derivatives of Inverse Functions in General

Let f be differentiable and have an inverse on an interval I. Let x_0 be a point in I at which $f'(x_0) \neq 0$. Then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

where $y_0 = f(x_0)$.

Example

Let $f(x) = 3x + 4$. Find $f^{-1}(x)$ and $\left(f^{-1}\right)'\left(\frac{1}{3}\right)$.

3.10 Book Problems

7-33 (odds), 37-41 (odds)

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8 7-11 March

- **Monday 7 March**

§3.9 Derivatives of Logarithmic and Exponential Functions
- Derivative of $y = \ln x$
- Derivative of $y = \ln |x|$
- Derivative of $y = b^x$
- Story Problem Example
- Derivatives of General Logarithmic Functions
- Neat Trick: Logarithmic Differentiation
- Book Problems

Midterm Review
- Running Out of Time on the Exam Plus other Study Tips
- Other Study Tips
- **Friday 11 March**

§3.10 Derivatives of Inverse Trigonometric Functions
- Derivative of Inverse Sine
- Derivative of Inverse Tangent
- Derivative of Inverse Secant
- All Other Inverse Trig Derivatives
- Derivatives of Inverse Functions in General
- Book Problems

§3.11 Related Rates

9 14-18 March

10 28 Mar – 1 April

11 4-8 April

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§3.11 Related Rates

In this section, we use our knowledge of derivatives to examine how variables change with respect to time.

The prime feature of these problems is that two or more variables, which are related in a known way, are themselves changing in time.

The goal of these types of problems is to determine the rate of change (i.e., the derivative) of one or more variables at a specific moment in time.

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Problem

The edges of a cube increase at a rate of 2 cm/sec. How fast is the volume changing when the length of each edge is 50 cm?

- **Variables:** V (Volume of the cube) and x (length of edge)
- **How Variables are related:** $V = x^3$
- **Rates Known:** $\frac{dx}{dt} = 2$ cm/sec
- **Rate We Seek:** $\frac{dV}{dt}$ when $x = 50$ cm

Note that both V and x are functions of t (their respective sizes are dependent upon how much time has passed).

So we can write $V(t) = x(t)^3$ and then differentiate this with respect to t:

$$V'(t) = 3x(t)^2 \cdot x'(t).$$

Note that $x(t)$ is the length of the cube's edges at time t, and $x'(t)$ is the rate at which the edges are changing at time t.

We can rewrite the previous equation as

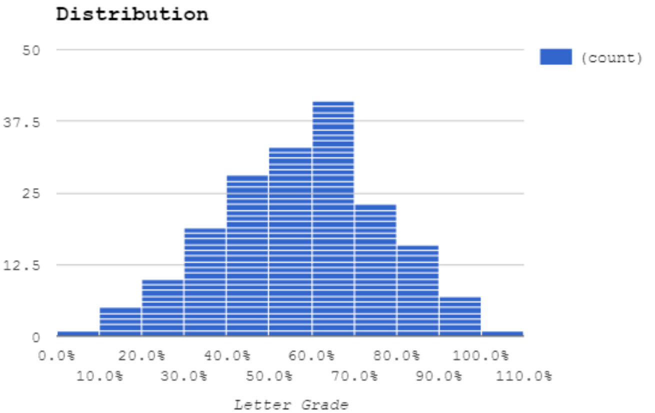
$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}.$$

So the rate of change of the volume when $x = 50$ cm is

$$\left.\frac{dV}{dt}\right|_{x=50} = 3 \cdot 50^2 \cdot 2 = 15000 \text{ cm}^3/\text{sec}.$$

π Day 2016

- Exam 2: Curve, etc. is posted.



π Day 2016 (cont.)

- Midterm: expect it back Thursday in drill. Don't expect a curve. :(
- "Fast Track Calculus": Dr. Kathleen Morris will be teaching a second 8 weeks Calculus One class. "If you have a student who is maybe doing poorly because of illness or a tragic event in their life during the beginning of the semester, this might be an opportunity for a new start for them." The class requires departmental consent so the student will need to contact Kathleen to get permission to enroll.

Steps for Solving Related Rates Problems

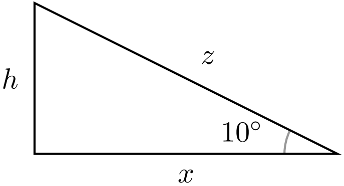
1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
2. Write one or more equations that express the basic relationships among the variables.
3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
4. Substitute known values and solve for the desired quantity.
5. Check that the units are consistent and the answer is reasonable.

The Jet Problem

A jet ascends at a 10° angle from the horizontal with an airspeed of 550 miles/hr (its speed along its line of flight is 550 miles/hr). How fast is the altitude of the jet increasing? If the sun is directly overhead, how fast is the shadow of the jet moving on the ground?

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

Step 1: There are three variables: the distance the shadow has traveled (x), the altitude of the jet (h), and the distance the jet has actually traveled on its line of flight (z). We know that $\frac{dz}{dt} = 550$ miles/hr and we want to find $\frac{dx}{dt}$ and $\frac{dh}{dt}$. We also see that these variables are related through a right triangle:



Step 2: To answer how fast the altitude is increasing, we need an equation involving only h and z. Using trigonometry,

$$\sin(10^\circ) = \frac{h}{z} \implies h = \sin(10^\circ) \cdot z.$$

To answer how fast the shadow is moving, we need an equation involving only x and z. Using trigonometry,

$$\cos(10^\circ) = \frac{x}{z} \implies x = \cos(10^\circ) \cdot z.$$

Step 3: We can now differentiate each equation to answer each question:

$$h = \sin(10^\circ) \cdot z \implies \frac{dh}{dt} = \sin(10^\circ)\frac{dz}{dt}$$
$$x = \cos(10^\circ) \cdot z \implies \frac{dx}{dt} = \cos(10^\circ)\frac{dz}{dt}$$

Step 4: We know that $\frac{dz}{dt} = 550$ miles/hr. So

$$\frac{dh}{dt} = \sin(10^\circ) \cdot 550 \approx 95.5 \text{ miles/hr}$$
$$\frac{dx}{dt} = \cos(10^\circ) \cdot 550 \approx 541.6 \text{ miles/hr}$$

Step 5: Because both answers are in terms of miles/hr and both answers seem reasonable within the context of the problem, we conclude that the jet is gaining altitude at a rate of 95.5 miles/hr, while the shadow on the ground is moving at about 541.6 miles/hr.

Example

The sides of a cube increase at a rate of R cm/sec. When the sides have a length of 2 cm, what is the rate of change of the volume?

Exercise

A 13 foot ladder is leaning against a vertical wall when Jack begins pulling the foot of the ladder away from the wall at a rate of 0.5 ft/sec. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 ft from the wall?

Exercise

Sand falls from an overhead bin and accumulates in a conical pile with a radius that is always three times its height. Suppose the height of the pile increases at a rate of 2 cm/sec. When the pile is 12 cm high, at what rate is the sand leaving the bin? *Recall the volume of a cone:* $V = \frac{1}{3}\pi r^2 h$.

3.11 Book Problems

5-14, 16-19, 21-24, 37-38

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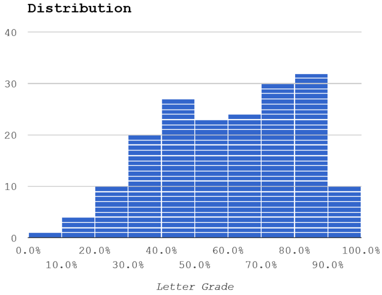
Wed 16 Mar

- Midterm: Take your raw score out of 140, instead of 150, for the curve.



- Office hours 1030-1230 today and Friday.

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§4.1 Maxima and Minima

Chapter 4 is all about applications of the derivative. In the first couple of sections we examine the graphs of functions and what the derivative can tell us about the graph's behavior and characteristics.

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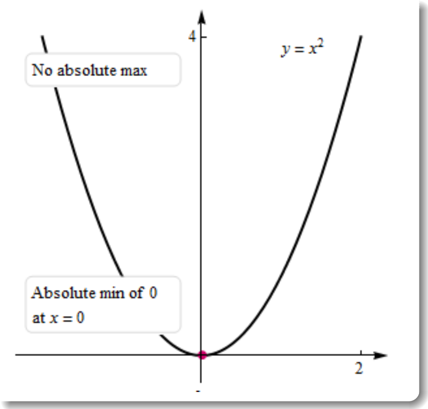
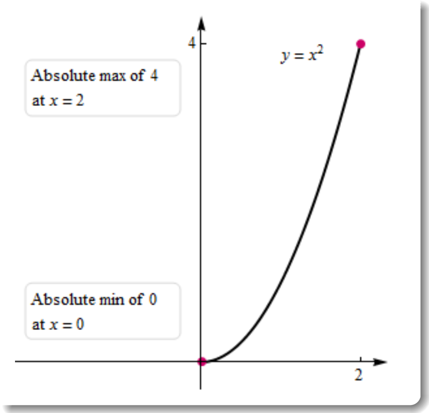
Definition

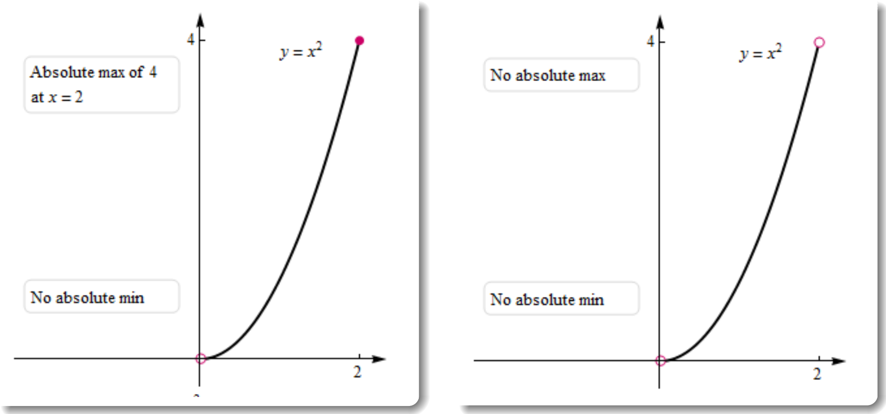
Let f be defined on an interval I containing c.

- f has an **absolute maximum** value on I at c means $f(c) \geq f(x)$ for every x in I.
- f has an **absolute minimum** value on I at c means $f(c) \leq f(x)$ for every x in I.

The existence and location of absolute extreme values depend on the function and the interval of interest:







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Extreme Value Theorem

Theorem (Extreme Value Theorem)

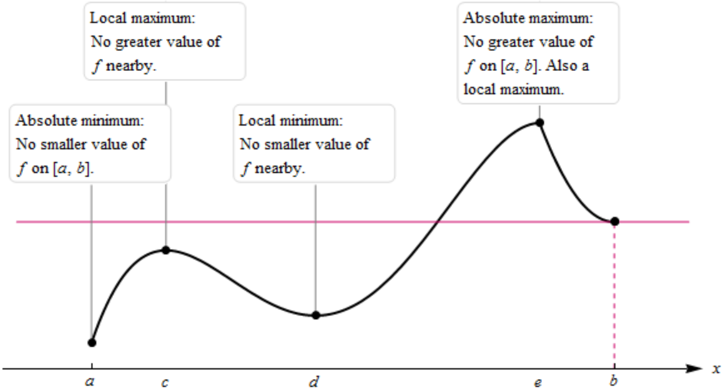
A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

The EVT provides the criteria that ensures absolute extrema:

- the function must be continuous on the interval of interest;
- the interval of interest must be closed and bounded.

Local Maxima and Minima

Beyond absolute extrema, a graph may have a number of peaks and dips throughout its interval of interest:



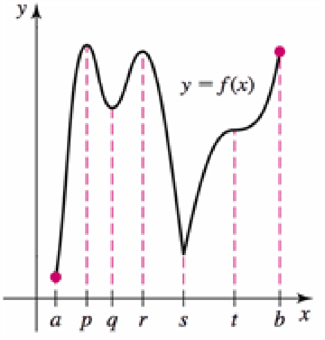
Definition

Suppose I is an interval on which f is defined and c is an interior point of I.

- If $f(c) \geq f(x)$ for all x in some open interval containing c, then $f(c)$ is a **local maximum** value of f.
- If $f(c) \leq f(x)$ for all x in some open interval containing c, then $f(c)$ is a **local minimum** value of f.

Exercise

Use the graph below to identify the points on the interval $[a, b]$ at which local and absolute extreme values occur.



Critical Points

Based on the previous graph, how is the derivative related to where the local extrema occur?

Local extrema occur where the derivative either does not exist or is equal to 0.

Definition

An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a **critical point** of f.

Local Extreme Point Theorem

Theorem (Local Extreme Point Theorem)

If f has a local minimum or maximum value at c and $f'(c)$ exists, then $f'(c) = 0$. **(Converse is not true!)**

It is possible for $f'(c) = 0$ or $f'(c)$ not to exist at a point, yet the point not be a local min or max. Therefore, critical points provide **candidates** for local extrema, but do not guarantee that the points are local extrema (see p. 227 immediately before Figure 4.9 for examples).

Locating Absolute Min and Max

Two facts help us in the search for absolute extrema:

- Absolute extrema in the interior of an interval are also local extrema, which occur at critical points of f.
- Absolute extrema may occur at the endpoints of f.

Procedure: Assume that the function f is continuous on $[a, b]$.

1. Locate the critical points c in (a, b), where $f'(c) = 0$ or $f'(c)$ does not exist. These points are **candidates** for absolute extrema.
2. Evaluate f at the critical points and at the endpoints of $[a, b]$.
3. Choose the largest and smallest values of f from Step 2 for the absolute max and min values, respectively.

NOTE: In this section, given an equation, we can identify critical points and absolute extrema, BUT NOT LOCAL EXTREMA. Techniques for locating local extrema come in later sections.

Example

On the interval $[-2, 2]$, the function $f(x) = x^4$

A. has no local or absolute extrema.

B. has a local minimum but no absolute minimum.

C. has an absolute maximum but no local maxima.

D. has an absolute maximum at an interior point of the interval.

Exercise

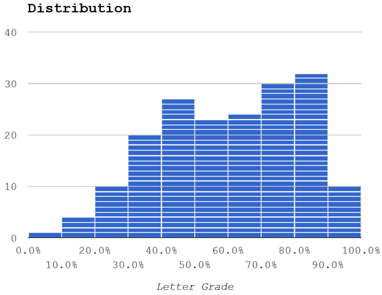
Given $f(x) = (x+1)^{4/3}$ on $[-8, 8]$, determine the critical points and the absolute extreme values of f.

4.1 Book Problems

11–35 (odds), 37–49 (odds)

Fri 18 Mar

- Midterm: Take your raw score out of 140, instead of 150, for the curve.



- Office hours 1030-1230 today.

8 7-11 March

9 14-18 March

- Monday 14 March
- Steps for Solving Related Rates Problems
- Book Problems
- Wednesday 16 March

§4.1 Maxima and Minima

- Extreme Value Theorem
- Local Maxima and Minima
- Critical Points
- Local Extreme Point Theorem
- Locating Absolute Min and Max
- Book Problems
- Friday 18 March

§4.2 What Derivatives Tell Us

- How is it related to the derivative?
- First Derivative Test
- Absolute extremes on any interval
- Derivative of the derivative tells us:
- Test for Concavity
- Second Derivative Test
- Book Problems

10 28 Mar – 1 April

11 4-8 April

§4.2 What Derivatives Tell Us

Definition

Suppose a function f is defined on an interval I.

- We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.
- We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

How is it related to the derivative?

Suppose f is continuous on an interval I and differentiable at every interior point of I.

- If $f'(x) > 0$ for all interior points of I, then f is increasing on I.
- If $f'(x) < 0$ for all interior points of I, then f is decreasing on I.

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Example

Sketch a function that is continuous on $(-\infty, \infty)$ that has the following properties:

- $f'(-1)$ is undefined;
- $f'(x) > 0$ on $(-\infty, -1)$;
- $f'(x) < 0$ on $(-1, 4)$;
- $f'(x) > 0$ on $(4, \infty)$.

Example

Find the intervals on which

$$f(x) = 3x^3 - 4x + 12$$

is increasing and decreasing. If you graph f and f' on the same axes, what do you notice?

First Derivative Test

Suppose that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c.
- If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c.
- If f' does not change sign at c (from positive to negative or vice versa), then f has **no** local extreme value at c.

NOTE: The First Derivative Test does NOT test for increasing/decreasing, only local max/min. Use it on critical points.

Exercise

If $f(x) = 2x^3 + 3x^2 - 12x + 1$, identify the critical points on the interval $[-3, 4]$, and use the First Derivative Test to locate the local maximum and minimum values. What are the absolute max and min?

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Absolute extremes on any interval

The Extreme Value Theorem (cf., Section 4.1) stated that we were guaranteed extreme values only on closed intervals.

However: Suppose f is continuous on an interval I that contains only one local extremum at $(x =)c$.

- If it is a local minimum, then $f(c)$ is the absolute minimum of f on I.
- If it is a local maximum, then $f(c)$ is the absolute maximum of f on I.

Derivative of the derivative tells us:

Just as the first derivative f' told us whether the function f was increasing or decreasing, the second derivative f'' also tells us whether f' is increasing or decreasing.

Definition

Let f be differentiable on an open interval I.

- If f' is increasing on I, then f is **concave up** on I.
- If f' is decreasing on I, then f is **concave down** on I.

Definition

If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an **inflection point** at c.

Test for Concavity

Suppose that f'' exists on an interval I.

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
- If c is a point of I at which $f''(c) = 0$ and f'' changes signs at c, then f has an inflection point at c.

Example

What would a function with the following properties look like?

1. $f' > 0$ and $f'' > 0$
2. $f' > 0$ and $f'' < 0$
3. $f' < 0$ and $f'' > 0$
4. $f' < 0$ and $f'' < 0$

Second Derivative Test

Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local maximum at c.
- If $f''(c) = 0$, then the test is inconclusive.

Exercise

Given $f(x) = 2x^3 - 6x^2 - 18x$

(a) Determine the intervals on which it is concave up or concave down, and identify any inflection points.

(b) Locate the critical points, and use the 2nd Derivative Test to determine whether they correspond to local minima or maxima, or whether the test is inconclusive.

4.2 Book Problems

11–47 (odds), 53–81 (odds)

Mon 28 Mar

- Exam 3: next week, probably Friday. Covers §3.10-4.6

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8 7-11 March

9 14-18 March

10 28 Mar – 1 April

- Monday 28 March

§4.3 Graphing Functions

- Book Problems

- **Wednesday 30 March**

§4.4 Optimization Problems

- Essential Feature of Optimization Problems
- Guidelines for Optimization Problems
- Book Problems
- **Friday 1 April**

§4.5 Linear Approximation and Differentials

- Linear Approximation
- Intro to Differentials
- Book Problems

11 4-8 April

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§4.3 Graphing Functions

Graphing Guidelines:

1. Identify the domain or interval of interest.
2. Exploit symmetry.
3. Find the first and second derivatives.
4. Find critical points and possible inflection points.
5. Find intervals on which the function is increasing or decreasing, and concave up/down.
6. Identify extreme values and inflection points.
7. Locate vertical/horizontal asymptotes and determine end behavior.
8. Find the intercepts.
9. Choose an appropriate graphing window and make a graph.

Exercise

According to the graphing guidelines, sketch a graph of

$$f(x) = \frac{x^2}{x^2 - 4}.$$

4.3 Book Problems

7,8, 15-35 (odds), 45-53

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Wed 30 Mar

- Exam 3: next Friday. Covers §3.10-4.6
- Algebra Seminar: today at 3p in SCEN 322.
  "The talk will be given by our own Ashley Wheeler on the Title: Local cohomology of Stanley-Reisner rings." (from the department email).

8 7-11 March

9 14-18 March

10 28 Mar – 1 April

- Monday 28 March

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11 4-8 April

§4.4 Optimization Problems

In many scenarios, it is important to find a maximum or minimum value under given constraints. Given our use of derivatives from the previous sections, optimization problems follow directly from what we have studied.

Question

Given all nonnegative real numbers x and y between 0 and 50 such that their sum is 50 (i.e., $x + y = 50$), which pair has the maximum product?

This is a basic optimization problem. In this problem, we are given a **constraint** ($x + y = 50$) and asked to maximize an **objective function** ($A = xy$).

The first step is to express the objective function $A = xy$ in terms of a single variable by using the constraint:

$$y = 50 - x \implies A(x) = x(50 - x).$$

To maximize A, we find the critical points:

$A'(x) = 50 - 2x$ which has a critical point at $x = 25$.

Since $A(x)$ has domain $[0, 50]$, to maximize A we evaluate A at the endpoints of the domain and at the critical point:

$$A(0) = A(50) = 0 \text{ and } A(25) = 625.$$

So 625 is the maximum value of A and A is maximized when $x = 25$ (which means $y = 25$).

Essential Feature of Optimization Problems

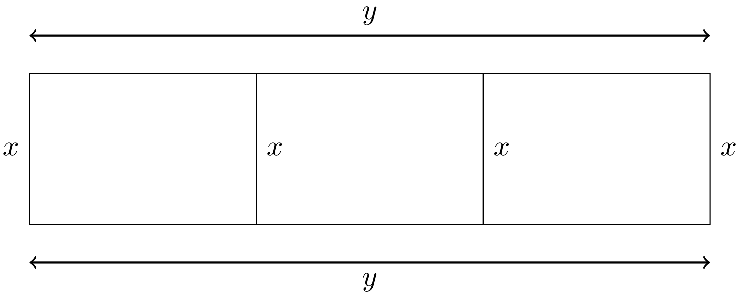
All optimization problems take the following form:

What is the maximum (or minimum) value of an objective function subject to the given constraint(s)?

Most optimization problems have the same basic structure as the previous problem: An objective function (possibly with several variables and/or constraints) with methods of calculus used to find the maximum/minimum values.

Exercise

Suppose you wish to build a rectangular pen with two interior parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?



By the picture, $2y + 4x = 500$ which implies $y = -2x + 250$. We are maximizing $A = xy$. So write

$$A(x) = x(-2x + 250) = -2x^2 + 250x.$$

Taking the derivative, $A'(x) = -4x + 250 = 0$, A has a critical point at $x = 62.5$.

From the picture, since we have 500 ft of fencing available we must have $0 \leq x \leq 125$. To find the max we must examine the points $x = 0, 62.5, 125$:

$$A(0) = A(125) = 0 \text{ and } A(62.5) = 7812.5$$

We see that

$$\boxed{\text{the maximum area is } 7812.5 \text{ ft}^2.}$$

The pen's dimensions (answer the question!) are $\boxed{x = 62.5 \text{ ft}}$ and

$$\boxed{y = -2(62.5) + 250 = 125 \text{ ft.}}$$

Guidelines for Optimization Problems

1. READ THE PROBLEM carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (i.e., the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

Question

The sum of a pair of positive real numbers that have a product of 9 is

$$S(x) = x + \frac{9}{x},$$

where x is one of the numbers. This sum $S(x)$ has a minimum when:

A. $x = 9$

B. $x = 3$

C. $x = 6$

D. none of the above

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Exercise

An open rectangular box with square base is to be made from 48 ft^2 of material. What dimensions will result in a box with the largest possible volume?

Exercise

Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and the graph of $y = 8 - x^3$.

4.4 Book Problems

5-16, 19-20, 24, 26

Fri 1 Apr

- Exam 3: next Friday. Covers §3.10-4.6

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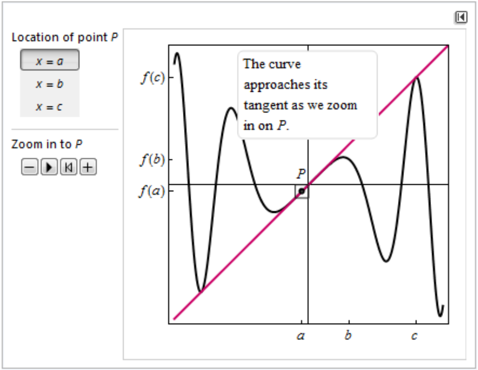
§4.5 Linear Approximation and Differentials

Suppose f is a function such that f' exists at some point P. If you zoom in on the graph, the curve appears more and more like the tangent line to f at P.



Linear Approximation

This idea – that smooth curves (i.e., curves without corners) appear straighter on smaller scales – is the basis of linear approximations.

One of the properties of a function that is differentiable at a point P is that it is locally linear near P (i.e., the curve approaches the tangent line at P.)

Therefore, it makes sense to approximate a function with its tangent line, which matches the value and slope of the function at P.

This is why you've had to do so many "find the equation for the tangent line to the given point" problems!

Definition

Suppose f is differentiable on an interval I containing the point a. The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a) \qquad \text{for } x \text{ in } I.$$

Remarks: Compare this definition to the following: At a given point $P = (a, f(a))$, the slope of the line tangent to the curve at P is $f'(a)$. So the equation of the tangent line is

$$y - f(a) = f'(a)(x - a).$$

(Yes, it is the same thing!)

Exercise

Write the equation of the line that represents the linear approximation to

$$f(x) = \frac{x}{x+1} \qquad \text{at } a = 1.$$

Then *use* the linear approximation to estimate $f(1.1)$.

Solution: First compute

$$f'(x) = \frac{1}{(x+1)^2}, \quad f(a) = \frac{1}{2}, \quad f'(a) = \frac{1}{4}$$

$$L(x) = \frac{1}{2} + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{1}{4}.$$

Solution (continued):

Because $x = 1.1$ is near $a = 1$, we can estimate $f(1.1)$ using $L(1.1)$:

$$f(1.1) \approx L(1.1) = 0.525$$

Note that $f(1.1) = 0.5238$, so the error in this estimation is

$$\frac{0.525 - 0.5238}{0.5238} \times 100 = 0.23\%.$$

Exercise

(a) The linear approximatioln to $f(x) = \sqrt{1+x}$ at the point $x = 0$ is (choose one):

A. $L(x) = 1$
B. $L(x) = 1 + \frac{x}{2}$
C. $L(x) = x$
D. $L(x) = 1 - \frac{x}{2}$

(b) What is an approximation for $f(0.1)$?

Intro to Differentials

Our linear approximation $L(x)$ is used to approximate $f(x)$ when a is fixed and x is a nearby point:

$$f(x) \approx f(a) + f'(a)(x - a)$$

When rewritten,

$$f(x) - f(a) \approx f'(a)(x - a)$$

$$\implies \Delta y \approx f'(a)\Delta x.$$

A change in y can be approximated by the corresponding change in x, magnified or diminished by a factor of $f'(a)$.

This is another way to say that $f'(a)$ is the rate of change of y with respect to x!

$$\Delta y \approx f'(a)\Delta x$$

$$\frac{\Delta y}{\Delta x} \approx f'(a)$$

So if f is differentiable on an interval I containing the point a, then the change in the value of f (the Δy), between two points a and $a + \Delta x$ in I, is approximately $f'(x)\Delta x$.

We now have two different, but related quantities:

- The change in the function $y = f(x)$ as x changes from a to $a + \Delta x$ (which we call Δy).
- The change in the linear approximation $y = L(x)$ as x changes from a to $a + \Delta x$ (called the differential, dy).

$$\Delta y \approx dy$$

When the x-coordinate changes from a to $a + \Delta x$:

- The function change is **exactly** $\Delta y = f(a + \Delta x) - f(a)$.
- The linear approximation change is

$$\begin{aligned}\Delta L &= L(a + \Delta x) - L(a) \\ &= \big(f(a) + f'(a)(a + \Delta x - a)\big) - \big(f(a) + f'(a)(a - a)\big) \\ &= f'(a)\Delta x\end{aligned}$$

and this is dy.

We define the differentials dx and dy to distinguish between the change in the function (Δy) and the change in the linear approximation (ΔL):

- dx is simply the change in x, i.e. Δx.
- dy is the change in the linear approximation, which is $\Delta L = f'(a)\Delta x$.

SO:

$$\Delta L = f'(a)\Delta x$$

$$dy = f'(a)dx$$

$$\frac{dy}{dx} = f'(a) \quad \text{(at } x = a\text{)}$$

Definition

Let f be differentiable on an interval containing x.

- A small change in x is denoted by the **differential** dx.
- The corresponding change in $y = f(x)$ is approximated by the **differential** $dy = f'(x)dx$; that is,

$$\Delta y = f(x + \Delta x) - f(x)$$

$$\approx dy = f'(x)dx.$$

The use of differentials is critical as we approach integration.

Example

Use the notation of differentials $[dy = f'(x)dx]$ to approximate the change in $f(x) = x - x^3$ given a small change dx.

Solution: $f'(x) = 1 - 3x^2$, so $dy = (1 - 3x^2)dx$.
A small change dx in the variable x produces an approximate change of $dy = (1 - 3x^2)dx$ in y.

For example, if x increases from 2 to 2.1, then $dx = 0.1$ and

$$dy = \left(1 - 3(2)^2\right)(0.1) = -1.1.$$

This means as x increases by 0.1, y decreases by 1.1.

4.5 Book Problems

13-20, 35-50

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

Mon 4 Apr

- Exam 3: Friday. Covers §3.10-4.6

10 28 Mar – 1 April

8 7-11 March

11 4-8 April

- Monday 4 April

9 14-18 March

§4.6 Mean Value Theorem

- Consequences of MVT
- Book Problems
- **Wednesday 6 April**

Exam #3 Review

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§4.6 Mean Value Theorem

In this section, we examine the Mean Value Theorem, one of the "big ideas" that provides the basis for much of calculus.

Before we get to the mean Value Theorem, we examine Rolle's Theorem, where the property $f(a) = f(b)$ holds, for some function $f(x)$ defined on an interval $[a, b]$.

Question

If you have two points $(a, f(a))$ and $(b, f(b))$, with the property that $f(a) = f(b)$, what might this look like?

Theorem (Rolle's Theorem)

Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.

Essentially what Rolle's Theorem concludes is that at some point(s) between a and b, f has a horizontal tangent.

Question

Note the hypotheses in this theorem: f is continuous on $[a, b]$ and differentiable on (a, b). Why are these important?

Exercise

Determine whether Rolle's Theorem applies to the function $f(x) = x^3 - 2x^2 - 8x$ on the interval $[-2, 0]$.

- If it doesn't, find an interval for which Rolle's Thm could apply to that function.
- If it does, what is the "c" value so that $f'(c) = 0$?

Theorem (Mean Value Theorem (MVT))

If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

See Figure 4.68 on p. 276 for a visual justification of MVT.

The slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ is

$$\frac{f(b) - f(a)}{b - a}.$$

MVT says that there is a point c on f where the tangent line at c (whose slope is $f'(c)$) is parallel to this secant line.

Question

Suppose you leave Fayetteville for a location in Fort Smith that is 60 miles away. If it takes you 1 hour to get there, what can we say about your speed? If it takes you 45 minutes to get there, what can we say about your speed?

Example

Let $f(x) = x^2 - 4x + 3$.

1. Determine whether the MVT applies to $f(x)$ on the interval $[-2, 3]$.
2. If so, find the point(s) that are guaranteed to exist by the MVT.

Example

How many points c satisfy the conclusion of the MVT for $f(x) = x^3$ on the interval $[-1, 1]$? Justify your answer.

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Consequences of MVT

Theorem (Zero Derivative Implies Constant Function)

If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

Theorem (Functions with Equal Derivatives Differ by a Constant)

If two functions have the property that $f'(x) = g'(x)$ for all x of an interval I, then $f(x) - g(x) = C$ on I, where C is a constant.

Theorem (Intervals of Increase and Decrease)

Suppose f is continuous on an interval I and differentiable at all interior points of I.

- *If $f'(x) > 0$ at all interior points of I, then f is increasing on I.*
- *If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.*

4.6 Book Problems

7-14, 17-24

Wed 6 Apr

- Exam 3: Friday. Covers §3.10-4.6. You will need a scientific calculator.

10 28 Mar – 1 April

8 7-11 March

11 4-8 April

- Monday 4 April

§4.6 Mean Value Theorem

- Consequences of MVT
- Book Problems
- Wednesday 6 April

Exam #3 Review

9 14-18 March

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Exam # 3 Review

- §3.10 Derivatives of Inverse Trig Functions
  - Know the derivatives of the six inverse trig functions.
  - Also: You are responsible for every derivative rule and every derivative formula we have covered this semester.
- §3.11 Related Rates
  - Know the steps to solving related rates problems, and be able to use them to solve problems given variables and rates of change.
  - Be able to solve related rates problems. If, while doing the HW (paper or computer), you were provided a formula in order to solve the problem, then I will do the same. If you were not provided a formula while doing the HW (paper or computer), then I also will not provide the formula.

Exam # 3 Review (cont.)

Exercise

An inverted conical water tank with a height of 12 ft and a radius of 6 ft is drained through a hole in the vertex at a rate of 2 ft^3/sec. What is the rate of change of the water depth when the water depth is 3 ft?

- §4.1 Maxima and Minima
  - Know the definitions of maxima, minima, and what makes these points local or absolute extrema (both analytically and graphically).
  - Know how to find critical points for a function.
  - Given a function on a given interval, be able to find local and/or absolute extrema.

Exam # 3 Review (cont.)

- Given specified properties of a function, be able to sketch a graph of that function.

- §4.2 What Derivatives Tell Us
  - Be able to use the first derivative to determine where a function is increasing or decreasing.
  - Be able to use the First Derivative Test to identify local maxima and minima. Be able to explain in words how you arrived at your conclusion.
  - Be able to find critical points, absolute extrema, and inflection points for a function.
  - Be able to use the second derivative to determine the concavity of a function.

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Exam # 3 Review (cont.)

- Be able to use the Second Derivative Test to determine whether a given point is a local max or min. Be able to explain in words how you arrived at your conclusion.
- Know your Derivative Properties!!! (see Figure 4.36 on p. 256)

- §4.3 Graphing Functions
  - Be able to find specific characteristics of a function that are spelled out in the Graphing Guidelines on p. 261 (e.g., know how to find x- and y-intercepts, vertical/horizontal asymptotes, critical points, inflection points, intervals of concavity and increasing/decreasing, etc.).
  - Be able to use these specific characteristics of a function to sketch a graph of the function.

Exam # 3 Review (cont.)

- §4.4 Optimization Problems
  - Be able to solve optimization problems that maximize or minimize a given quantity.
  - Be able to identify and express the constraints and objective function in an optimization problem.
  - Be able to determine your interval of interest in an optimization problem (e.g., what range of x-values are you searching for your extreme points?)
  - **As to formulas, the same comment made above with respect to formulas for related rates problems applies here as well.**

Exam # 3 Review (cont.)

Exercise

What two nonnegative real numbers a and b whose sum is 23 will

(a) minimize $a^2 + b^2$?

(b) maximize $a^2 + b^2$?

- §4.5 Linear Approximation and Differentials
  - Be able to find a linear approximation for a given function.
  - Be able to use a linear approximation to estimate the value of a function at a given point.
  - Be able to use differentials to express how the change in x (dx) impacts the change in y (dy).

Exam # 3 Review (cont.)

- §4.6 Mean Value Theorem (for Derivatives)
  - Know and be able to state Rolle's Thm and the Mean Value Thm, including knowing the hypotheses and conclusions for both.
  - Be able to apply Rolle's Thm to find a point in a given interval.
  - Be able to apply the MVT to find a point in a given interval.
  - Be able to use the MVT to find equations of secant and tangent lines.

Exam # 3 Review (cont.)

Exercise (s)

Determine whether the Mean Value Theorem (or Rolle's Theorem) applies to the following functions. If it does, then find the point(s) guaranteed by the theorem to exist.

(1) $f(x) = \sin(2x)$ on $\left[0, \frac{\pi}{2}\right]$

(2) $g(x) = \ln(2x)$ on $[1, e]$

(3) $h(x) = 1 - |x|$ on $[-1, 1]$

Exam # 3 Review (cont.)

Exercise (s)

(4) $j(x) = x + \frac{1}{x}$ on $[1, 3]$

(5) $k(x) = \frac{x}{x+2}$ on $[-1, 2]$

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Part 4. Introduction to Integrals

12 11-15 April

- **Monday 11 April**

§4.7 L'Hôpital's Rule

- L'Hôpital's Rule in disguise
- Other Indeterminate Forms
- **Wednesday 13 April**
- Examining Growth Rates
- Pitfalls in Using Lôpital's Rule
- Book Problems

§4.9 Antiderivatives

- Indefinite Integrals
- Rules for Indefinite Integrals
- Indefinite Integrals of Trig Functions
- Other Indefinite Integrals
- **Friday 15 April**
- Initial Value Problems
- Book Problems

§5.1 Approximating Area Under Curves

- Riemann Sums

13 18-22 April

- **Monday 18 April**
- Sigma Notation
- Σ-Shortcuts

Part 4. Introduction to Integrals (cont.)

Part 4. Introduction to Integrals (cont.)

14 25-29 April

- **Monday 25 April**

Exam #4 Review

- Other Remarks on the Exam
- **Wednesday 27 April**

Exam #4 Review

- Other Remarks on the Exam

15 2-4 May

- **Monday 2 May**

§5.5 Substitution Rule

- Integration by Trial and Error
- Substitution Rule
- Substitution Rule for Indefinite Integrals
- Procedure for Substitution Rule (Change of Variables)
- Variations on Substitution Rule
- Substitution Rule for Definite Integrals
- Book Problems
- **Wednesday 4 May**

Final Preparation

- About the Test
- Advice for the FINAL
- Easter Egg-xercises

Mon 11 Apr

- Exam 3: expect Thursday in drill. Stay tuned for the solutions. Protocol for appeals.
- No (scheduled) office hours Friday.

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§4.7 L'Hôpital's Rule

In Ch. 2, we examined limits that were computed using analytical techniques. Some of these limits, in particular those that were indeterminate, could not be computed with simple analytical methods.

For example,

$$\lim_{x\to 0}\frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 0}\frac{1-\cos x}{x}$$

are both limits that can't be computed by substitution, because plugging in 0 for x gives $\frac{0}{0}$.

Theorem (L'Hôpital's Rule $\left(\frac{0}{0}\right)$)

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Example

Evaluate the following limit:

$$\lim_{x\to -1} \frac{x^4+x^3+2x+2}{x+1}.$$

Solution: By direct substitution, we obtain 0/0. So we must apply l'Hôpital's Rule (LR) to evaluate the limit:

$$\begin{aligned}
\lim_{x\to -1} \frac{x^4+x^3+2x+2}{x+1} &\overset{\text{LR}}{=} \lim_{x\to -1} \frac{\dfrac{d}{dx}\left(x^4+x^3+2x+2\right)}{\dfrac{d}{dx}(x+1)} \\
&= \lim_{x\to -1} \frac{4x^3+3x^2+2}{1} \\
&= -4+3+2 = 1
\end{aligned}$$

Theorem (L'Hôpital's Rule ($\frac{\infty}{\infty}$))

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$).

(The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.)

Exercise

Evaluate the following limits using l'Hôpital's Rule:

- $$\lim_{x\to\infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$$
- $$\lim_{x\to 0} \frac{\tan 4x}{\tan 7x}$$

L'Hôpital's Rule in disguise

Other indeterminate limits in the form $0 \cdot \infty$ or $\infty - \infty$ cannot be evaluated directly using l'Hôpital's Rule.

For $0 \cdot \infty$ cases, we must rewrite the limit in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. A common technique is to divide by the reciprocal:

$$\lim_{x\to\infty} x^2 \sin\left(\frac{1}{5x^2}\right) = \lim_{x\to\infty} \frac{\sin\left(\dfrac{1}{5x^2}\right)}{\dfrac{1}{x^2}}$$

Exercise

Compute $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$.

For $\infty - \infty$, we can divide by the reciprocal as well as use a change of variables:

Example

Find $\lim\limits_{x\to\infty} x - \sqrt{x^2 + 2x}$.

Solution:

$$\begin{aligned}
\lim_{x\to\infty} x - \sqrt{x^2+2x} &= \lim_{x\to\infty} x - \sqrt{x^2(1+\frac{2}{x})} \\
&= \lim_{x\to\infty} x - x\sqrt{1+\frac{2}{x}} \\
&= \lim_{x\to\infty} x\left(1 - \sqrt{1+\frac{2}{x}}\right) \\
&= \lim_{x\to\infty} \frac{1-\sqrt{1+\frac{2}{x}}}{\frac{1}{x}}
\end{aligned}$$

This is now in the form $\frac{0}{0}$, so we can apply l'Hôpital's Rule and evaluate the limit.

In this case, it may even help to change variables. Let $t = \frac{1}{x}$:

$$\lim_{x\to\infty} \frac{1-\sqrt{1+\frac{2}{x}}}{\dfrac{1}{x}} = \lim_{t\to 0^+} \frac{1-\sqrt{1+2t}}{t}.$$

Other Indeterminate Forms

Limits in the form 1^∞, 0^0, and ∞^0 are also considered indeterminate forms, and to use l'Hôpital's Rule, we must rewrite them in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Here's how:

Assume $\lim\limits_{x\to a} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0, or ∞^0.

1. Evaluate $L = \lim\limits_{x\to a} g(x) \ln f(x)$. This limit can often be put in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, which can be handled by l'Hôpital's Rule.
2. Then $\lim\limits_{x\to a} f(x)^{g(x)} = e^L$. Don't forget this step!

Example

Evaluate $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution: This is in the form 1^∞, so we need to examine

$$\begin{aligned} L &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\ &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\ &\overset{\text{LR}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1. \end{aligned}$$

NOT DONE! Write

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^L = e^1 = e.$$

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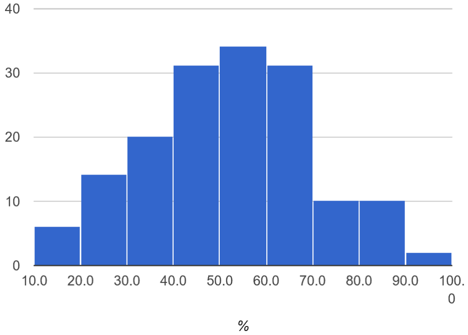
Wed 13 Apr

- Exam 3

		Problem																
	Total	1a	1b	1c	2	3a	3b	3c	4	5a	5b	5c	5d	5e	5f	5g	5h	5i
out of	75	5	3	3	12	3	3	5	12	2	2	4	3	3	3	3	4	4
Median-->	39	5	1	0	3	3	3	3	6	2	1	2	3	2	2	1	1	1.5

Spread:



Wed 13 Apr (cont.)

- No (scheduled) office hours today. I will be in ~1220p.
- ALL MLPs are open now.
- April 22: Last day to drop with a "W".

Examining Growth Rates

We can use l'Hôpital's Rule to examine the rate at which functions grow in comparison to one another.

Definition

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then $\boldsymbol{f}$ **grows faster than** $\boldsymbol{g}$ as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \text{ or } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

$g \ll f$ means that f grows faster than g as $x \to \infty$.

Definition

The functions f and g have **comparable growth rates** if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M, \text{ where } 0 < M < \infty.$$

Pitfalls in Using l'Hôpital's Rule

1. L'Hôpital's Rule says that $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. NOT

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left[\frac{f(x)}{g(x)}\right]' \text{ or } \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left[\frac{1}{g(x)}\right]' f'(x)$$

   (i.e., don't confuse this rule with the Quotient Rule).

2. Be sure that the limit with which you are working is in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

3. When using l'Hôpital's Rule more than once, simplify as much as possible before repeating the rule.

4. If you continue to use l'Hôpital's Rule in an unending cycle, another method must be used.

4.7 Book Problems

13-59 (odds), 69-79 (odds)

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§4.9 Antiderivatives

With differentiation, the goal of problems was to find the function f' given the function f.

With antidifferentiation, the goal is the opposite. Here, given a function f, we wish to find a function F such that the derivative of F is the given function f (i.e., $F' = f$).

Definition

A function F is called an **antiderivative** of a function f on an interval I provided $F'(x) = f(x)$ for all x in I.

Example

Given $f(x) = 4$, an antiderivative of $f(x)$ is $F(x) = 4x$.

NOTE: Antiderivatives are not unique!

They differ by a constant (C):

Theorem

Let F be any antiderivative of f. Then **all** *the antiderivatives of f have the form $F + C$, where C is an arbitrary constant.*

Recall: $\dfrac{d}{dx} f(x) = f'(x)$ is the derivative of $f(x)$.

Now: $\int f(x)\, dx = F + C$ *is* the antiderivative of $f(x)$. It doesn't matter which F you choose, since writing the C will show you are talking about all the antiderivatives at once. The C is also why we call it the *indefinite* integral.

Example

Find the antiderivatives of the following functions:

(1) $f(x) = -6x^{-7}$

(2) $g(x) = -4\cos 4x$

(3) $h(x) = \csc^2 x$

Indefinite Integrals

Example

$\int 4x^3\ dx = x^4 + C$, where C is the **constant of integration**.

The dx is called the **differential** and it is the same dx from Section 4.5. Like the $\frac{d}{dx}$, it shows which variable you are talking about. The function written between the $\int$ and the dx is called the **integrand**.

Rules for Indefinite Integrals

Power Rule: $\int x^p \, dx = \dfrac{x^{p+1}}{p+1} + C$

(p is any real number except -1)

Constant Multiple Rule: $\int cf(x) \, dx = c \int f(x) \, dx$

Sum Rule: $\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$

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Exercise

$$\int (5x^4 + 2x + 1)\, dx =$$

A. $20x^3 + 2 + C$

B. $x^5 + x^2 - x + C$

C. $x^5 + x^2 + C$

D. $x^5 + 2x^2 - x + C$

Exercise

Evaluate the following indefinite integrals:

(1) $\int \left(3x^{-2} - 4x^2 + 1\right)\ dx$

(2) $\int 6\sqrt[3]{x}\ dx$

(3) $\int 2\cos(2x)\ dx$

Indefinite Integrals of Trig Functions

Table 4.9 (p. 322) provides us with rules for finding indefinite integrals of trig functions.

1. $\frac{d}{dx}(\sin ax) = a\cos ax \longrightarrow \int \cos ax\ dx = \frac{1}{a}\sin ax + C$
2. $\frac{d}{dx}(\cos ax) = -a\sin ax \longrightarrow \int \sin ax\ dx = -\frac{1}{a}\cos ax + C$
3. $\frac{d}{dx}(\tan ax) = a\sec^2 ax \longrightarrow \int \sec^2 ax\ dx = \frac{1}{a}\tan ax + C$
4. $\frac{d}{dx}(\cot ax) = -a\csc^2 ax \longrightarrow \int \csc^2 ax\ dx = -\frac{1}{a}\cot ax + C$
5. $\frac{d}{dx}(\sec ax) = a\sec ax\tan ax \longrightarrow \int \sec ax\tan ax\ dx = \frac{1}{a}\sec ax + C$
6. $\frac{d}{dx}(\csc ax) = -a\csc ax\cot ax \longrightarrow \int \csc ax\cot ax\ dx = -\frac{1}{a}\csc ax + C$

Example

Evaluate the following indefinite integral: $\int 2\sec^2 2x \; dx$.

Solution: Using rule 3, with $a = 2$, we have

$$\int 2\sec^2 2x \; dx = 2\int \sec^2 2x \; dx = 2\left[\frac{1}{2}\tan 2x\right] + C = \tan 2x + C.$$

Exercise

Evaluate $\int 2\cos(2x) \; dx$.

Other Indefinite Integrals

7. $\dfrac{d}{dx}(e^{ax}) = ae^{ax} \longrightarrow \displaystyle\int e^{ax}\ dx = \frac{1}{a}e^{ax} + C$

8. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x} \longrightarrow \displaystyle\int \frac{dx}{x} = \ln|x| + C$

9. $\dfrac{d}{dx}\left(\sin^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{1}{\sqrt{a^2 - x^2}} \longrightarrow \displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

10. $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{a}{a^2 + x^2} \longrightarrow \displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

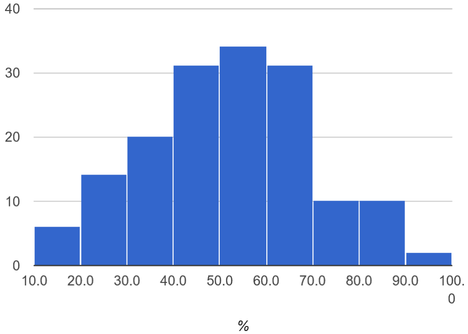
11. $\dfrac{d}{dx}\left(\sec^{-1}\left|\dfrac{x}{a}\right|\right) = \dfrac{a}{x\sqrt{x^2 - a^2}} \longrightarrow \displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

Fri 15 Apr

- Exam 3

	Total	Problem 1a	1b	1c	2	3a	3b	3c	4	5a	5b	5c	5d	5e	5f	5g	5h	5i
out of	75	5	3	3	12	3	3	5	12	2	2	4	3	3	3	3	4	4
Median-->	39	5	1	0	3	3	3	3	6	2	1	2	3	2	2	1	1	1.5

Spread:



Fri 15 Apr (cont.)

- No (scheduled) office hours today. I will be in ~1220p.
- ALL MLPs are open now.
- April 22: Last day to drop with a "W".

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Initial Value Problems

In some instances, you have enough information to determine the value of C in the antiderivative. These are often called **initial value problems**. Finding $f(x)$ is often called **finding the solution**.

Example

If $f'(x) = 7x^6 - 4x^3 + 12$ and $f(1) = 24$, find $f(x)$.

Solution: $f(x) = \int (7x^6 - 4x^3 + 12)\, dx = x^7 - x^4 + 12x + C$. Now find out which C gives $f(1) = 24$:

$$24 = f(1) = 1 - 1 + 12 + C,$$

so $C = 12$. Hence, $f(x) = x^7 - x^4 + 12x + 12$.

Exercise

Find the function f that satisfies $f''(t) = 6t$ with $f'(0) = 1$ and $f(0) = 2$.

4.9 Book Problems

11-45 (odds), 59-73 (odds), 83-93 (odds)

Advice: To solve 83-93 (odds), read pages 325-326, focusing on Example 8.

The base for these slides was done by Dr. Shannon Dingman, later encoded into LATEX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

§5.1 Approximating Area Under Curves

In the previous two chapters, we have come to see the derivative of a function associated with the rate of change of a function as well as the slope of the tangent line to the curve.

In the first section of Chapter 5, we now examine the meaning of the integral.

Question

If we know the velocity function of a particular object, what does that tell us about its position function?

Example

Suppose you ride your bike at a constant velocity of 8 miles per hour for 1.5 hours.

(a) What is the velocity function that models this scenario?

(b) What does the graph of the velocity function look like?

(c) What is the position function for this scenario?

(d) Where is the displacement (i.e., the distance you've traveled) represented when looking at the graph of the velocity function?

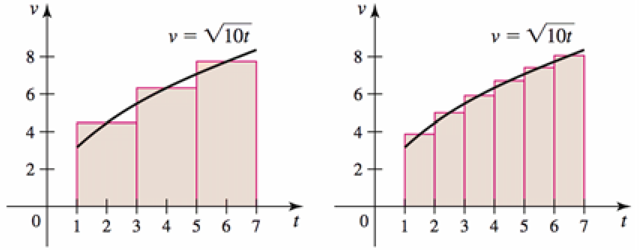
In the previous example, the velocity was constant. In most cases, this is not accurate (or possible). How could we find displacement when the velocity is changing over an interval?

One strategy is to divide the time interval into a particular number of subintervals and approximate the velocity on each subinterval with a constant velocity. Then for each subinterval, the displacement can be evaluate and summed.

Note: This provides us with only an approximation, but with a larger number of subintervals, the approximation becomes more accurate.

Example

Suppose the velocity of an object moving along a line is given by $v(t) = \sqrt{10t}$ on the interval $1 \le t \le 7$. Divide the time interval into $n = 3$ subintervals, assuming the object moves at a constant velocity equal to the value of v evaluated at the midpoint of the subinterval. Estimate the displacement of the object on $[1, 7]$. Repeat for $n = 6$ subintervals.



Riemann Sums

The more subintervals you divide your time interval into, the more accurate your approximation of displacement will be.We now examine a method for approximating areas under curves.

Consider a function f over the interval $[a, b]$. Divide $[a, b]$ into n subintervals of equal length:

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

with $x_0 = a$ and $x_n = b$. The length of each subinterval is denoted

$$\Delta x = \frac{b-a}{n}.$$

In each subinterval $[x_{k-1}, x_k]$ (where k is any number from 1 to n), we can choose any point $\overline{x}_k$ (note $\overline{x}_k$ might be different, depending on which k), and create a rectangle with a height of $f(\overline{x}_k)$.

The area of the rectangle is "base times height", written $f(\overline{x}_k)\Delta x$, since the base is the length of the subinterval.

Doing this for each subinterval, and then summing each rectangle's area, produces an approximation of the overall area. This approximation is called a **Riemann sum**

$$R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x.$$

The symbol "k" is what's known as an **indexing variable**. We let k vary from 1 to n, and we always have $x_{k-1} \leq \overline{x}_k \leq x_k$.

Note: We usually choose $\overline{x}_k$ so that it is consistent across all the subintervals.

Definition

Suppose

$$R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x$$

is a Riemann sum. Then:

1. R is a **left Riemann sum** when we choose $\overline{x}_k = x_{k-1}$ for each k (so $\overline{x}_k$ is the left endpoint of the subinterval).
2. R is a **right Riemann sum** when we choose $\overline{x}_k = x_k$ for each k (so $\overline{x}_k$ is the right endpoint of the subinterval).
3. R is a **midpoint Riemann sum** when we take $\overline{x}_k$ to be the midpoint between x_{k-1} and x_k, for each k.

(See pages 337-339 for picture of these.)

Example

Calculate the left Riemann sum for the function $f(x) = x^2 - 1$ on the interval $[2, 4]$ when $n = 4$.

A. 13.75

B. 19.75

C. 27.5

D. 55

Exercise

Compute the left, right, and midpoint Riemann sums for the function $f(x) = 2x^3$ on the interval $[0, 8]$ with $n = 4$.

Mon 18 Apr

- April 22: Last day to drop with a "W".
- Exam 4 next week, probably Friday. Covers §4.7-5.4

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Sigma Notation

Riemann sums become more accurate when we make n (the number of rectangles) bigger, but obviously writing it all down is no fun! Sigma notation gives a shorthand. Here is how sigma notation works, through an example:

Example

$\sum_{n=1}^{5} n^2$ is the sum all integer values from the lowest limit $(n = 1)$ to the highest limit $(n = 5)$ in the summand n^2 (in this case n is the indexing variable).

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

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Example

Evaluate $\sum_{k=0}^{3}(2k-1)$.

Solution: In this example, k is the indexing variable. It starts at 0 and goes to 3, which means we write down the expression in the parentheses for each of the integers from 0 to 3, then add the results together:

$$\sum_{k=0}^{3}(2k-1) = (2(0)-1)+(2(1)-1)+(2(2)-1)+(2(3)-1)$$
$$= -1+1+3+5 = 8.$$

Σ-Shortcuts

(n is always a positive integer)

$$\sum_{k=1}^{n} c = cn \text{ (where } c \text{ is a constant)}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Question

What is the indexing variable in these formulas?

Riemann Sums Using Sigma Notation

Suppose f is defined on a closed interval $[a, b]$ which is divided into n subintervals of equal length Δx. As before, $\overline{x}_k$ denotes a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$. Recall that $x_0 = a$ and $x_n = b$.

Here is how we can write the Riemann sum in a much more compact form:

$$R = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x = \sum_{k=1}^{n} f(\overline{x}_k)\Delta x.$$

With sigma-notation we can even derive explicit formulas for the basic Riemann sums (the expression in red is $\overline{x}_k$ for each case:

1. $\displaystyle\sum_{k=1}^{n} f(a+(k-1)\Delta x)\Delta x =$ left Riemann sum
2. $\displaystyle\sum_{k=1}^{n} f(a+k\Delta x)\Delta x =$ right Riemann sum
3. $\displaystyle\sum_{k=1}^{n} f(a+\left(k-\tfrac{1}{2}\right)\Delta x)\Delta x =$ midpoint Riemann sum

Exercise

(a) Use sigma notation to write the left, right, and midpoint Riemann sums for the function $f(x) = x^2$ on the interval $[1, 5]$ given that $n = 4$.

(b) Based on these approximations, estimate the area bounded by the graph of $f(x)$ over $[1, 5]$.

Suggestion: As n gets very big, Riemann sums, along with the Σ-shortcuts plus algebra, often make the problem way more manageable.

5.1 Book Problems

9-37

12 11-15 April

13 18-22 April

§5.2 Definite Integrals

- Net Area
- General Riemann Sums
- The *Definite* Integral
- Evaluating Definite Integrals

- Properties of Integrals
- Book Problems
- **Wednesday 20 April**
- Properties of Integrals
- Book Problems

14 25-29 April

15 2-4 May

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

§5.2 Definite Integrals

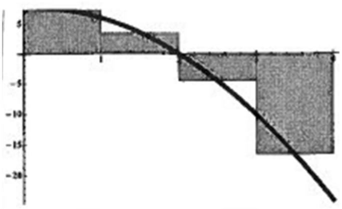
In §5.1, we saw how we can use Riemann sums to approximate the area under a curve. However, the curves we worked with were all non-negative.

Question

What happens when the curve is negative?

Example

Let $f(x) = 8 - 2x^2$ over the interval $[0, 4]$. Use a left, right, and midpoint Riemann sum with $n = 4$ to approximate the area under the curve.



Net Area

In the previous example, the areas where f was positive provided positive contributions to the area, while areas where f was negative provided negative contributions. The difference between positive and negative contributions is called the **net area**.

Definition

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis minus the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

The Riemann sums give approximations for the area under the curve. To make these approximations more and more accurate, we divide the region into more and more subintervals. To make these approximations exact, we allow the number of subintervals $n \to \infty$, thereby allowing the length of the subintervals $\Delta x \to 0$. In terms of limits:

$$\text{Net Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(\overline{x}_k)\Delta x.$$

General Riemann Sums

Suppose $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$ and let $\overline{x}_k$ be any point in $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$. If f is defined on $[a, b]$, then the sum

$$\sum_{k=1}^{n} f(\overline{x}_k)\Delta x_k = f(\overline{x}_1)\Delta x_1 + f(\overline{x}_2)\Delta x_2 + \cdots + f(\overline{x}_n)\Delta x_n$$

is called a **general Riemann sum for f on $[a, b]$**.

Note: In this definition, the lengths of the subintervals do not have to be equal.

The *Definite* Integral

As $n \to \infty$, all of the $\Delta x_k \to 0$, even the largest of these. Let Δ be the largest of the Δx_k's.

Definition

The **definite integral of f from a to b** is

$$\int_a^b f(x)\ dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(\overline{x}_k)\Delta x_k,$$

where f is a function defined on $[a, b]$. When this limit exists – over *all* partitions of $[a, b]$ and *all* choices of $\overline{x}_k$ on a partition – f is called **integrable**.

Evaluating Definite Integrals

Theorem

If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

See Figure 5.23, p. 325, for an example of a noncontinuous function that is integrable.

Knowing the limit of a Riemann sum, we can now translate that to a definite integral.

Example

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} (4\overline{x}_k - 3)\Delta x_k \text{ on } [-1, 4] \quad \equiv \quad \int_{-1}^{4} (4x - 3)\ dx$$

Without formally examining methods to evaluate definite integrals, we can use geometry.

Exercise

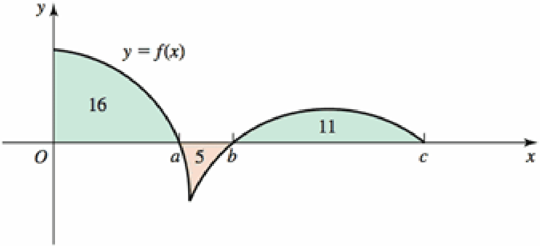
Using geometry, evaluate $\int_1^2 (4x-3)\ dx$.

(*Hint:* The area of a trapezoid is $A = \frac{h(l_1+l_2)}{2}$, where h is the height of the trapezoid and l_1 and l_2 are the lengths of the two parallel bases.)

Exercise

Using the picture below, evaluate the following definite integrals:

1. $\displaystyle\int_0^a f(x)\,dx$ 2. $\displaystyle\int_0^b f(x)\,dx$ 3. $\displaystyle\int_0^c f(x)\,dx$ 4. $\displaystyle\int_a^c f(x)\,dx$



Properties of Integrals

1. (Reversing Limits) $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$
2. (Identical Limits) $\int_a^a f(x)\ dx = 0$
3. (Integral of a Sum)
   $\int_a^b (f(x) + g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
4. (Constants in Integrals) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$

Properties of Integrals, cont.

5. (Integrals over Subintervals) If c lies between a and b, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

6. (Integrals of Absolute Values) The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\ dx$ is the sum of the areas of regions bounded by the graph of f and the x-axis on $[a, b]$. (See Figure 5.31 on p. 329)

   (This is the total area, no negative signs.)

Exercise

If $\int_2^4 f(x)\ dx = 3$ and $\int_4^6 f(x)\ dx = -2$, then compute $\int_2^6 f(x)\ dx$.

5.2 Book Problems

11-45 (odds), 67-74

Wed 20 Apr

- Exam 3: Issue with increasing/decreasing, number lines, etc. Point for signature.
- April 22: Last day to drop with a "W".
- Exam 4 next week, probably Friday. Covers §4.7-5.4

Without formally examining methods to evaluate definite integrals, we can use geometry.

Exercise

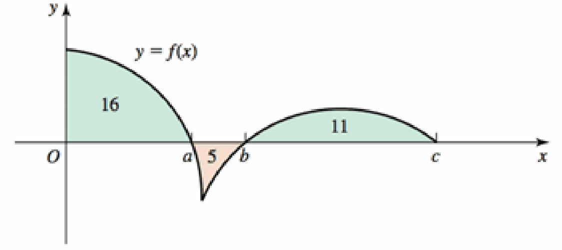
Using geometry, evaluate $\int_1^2 (4x-3)\ dx$.

(*Hint:* The area of a trapezoid is $A = \frac{h(l_1+l_2)}{2}$, where h is the height of the trapezoid and l_1 and l_2 are the lengths of the two parallel bases.)

Exercise

Using the picture below, evaluate the following definite integrals:

1. $\displaystyle\int_0^a f(x)\,dx$ 2. $\displaystyle\int_0^b f(x)\,dx$ 3. $\displaystyle\int_0^c f(x)\,dx$ 4. $\displaystyle\int_a^c f(x)\,dx$



Properties of Integrals

1. (Reversing Limits) $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$
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3. (Integral of a Sum)
   $\int_a^b (f(x) + g(x))\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
4. (Constants in Integrals) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$

Properties of Integrals, cont.

5. (Integrals over Subintervals) If c lies between a and b, then

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6. (Integrals of Absolute Values) The function $|f|$ is integrable on $[a,b]$ and $\int_a^b |f(x)|\ dx$ is the sum of the areas of regions bounded by the graph of f and the x-axis on $[a,b]$. (See Figure 5.31 on p. 329)

   (This is the total area, no negative signs.)

Exercise

If $\int_2^4 f(x)\ dx = 3$ and $\int_4^6 f(x)\ dx = -2$, then compute $\int_2^6 f(x)\ dx$.

5.2 Book Problems

11-45 (odds), 67-74

12 11-15 April

13 18-22 April

§5.3 Fundamental Theorem of Calculus

- Area Functions
- The Fundamental Theorem of Calculus (Part 1)
- The Fundamental Theorem of Calculus (Part 2)
- Overview of FTOC
- Book Problems
- **Friday 22 April**

14 25-29 April

15 2-4 May

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

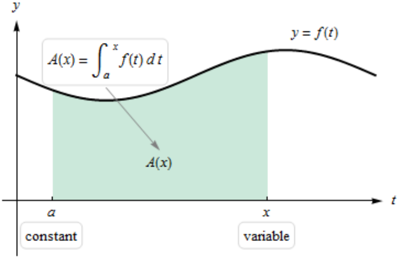
§5.3 Fundamental Theorem of Calculus

Using Riemann sums to evaluate definite integrals is usually neither efficient nor practical. We will develop methods to evaluate integrals and also tie together the concepts of differentiation and integration.

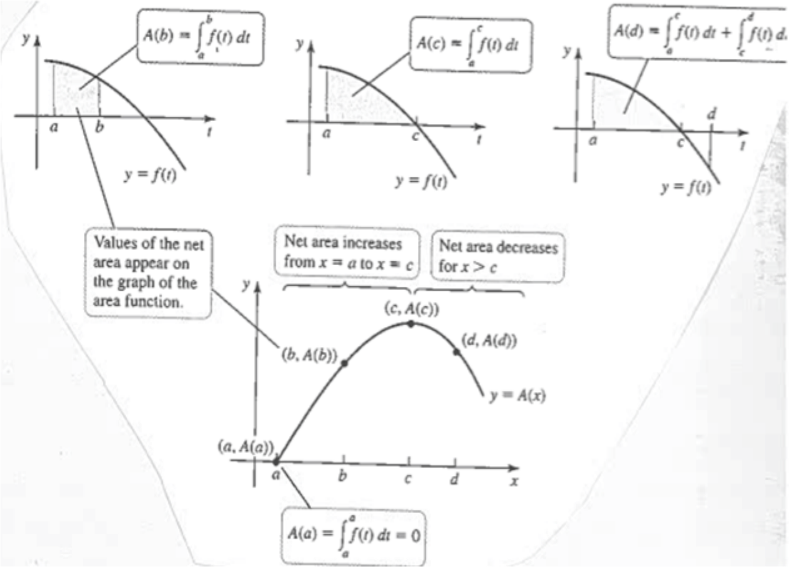
To connect the concepts of differention and integration, we first must define the concept of an area function.

Area Functions

Let $y = f(t)$ be a continuous function which is defined for all $t \geq a$, where a is a fixed number. The area function for f with left endpoint at a is given by $A(x) = \int_a^x f(t)\ dt$.



This gives the net area of the region between the graph of f and the t-axis between the points $t = a$ and $t = x$.



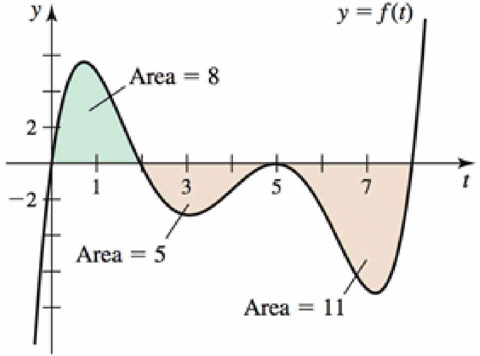
The base ... K. Wheeler.

Example

The graph of f is shown below. Let

$$A(x) = \int_0^x f(t)\ dt \quad \text{and} \quad F(x) = \int_2^x f(t)\ dt$$

be two area functions for f. Compute $A(2), F(5), A(5), F(8)$.



The Fundamental Theorem of Calculus (Part 1)

Linear functions help to build the rationale behind the Fundamental Theorem of Calculus.

Example

Let $f(t) = 4t + 3$ and define $A(x) = \int_1^x f(t)\ dt$. What is $A(2)$? $A(4)$? $A(x)$? $A'(x)$?

In general, the property illustrated with this linear function works for all continuous functions and is one part of the FTOC (Fundamental Theorem of Calculus).

Theorem (FTOC I)

If f is continuous on $[a, b]$, then the area function $A(x) = \int_a^x f(t)\ dt$ for $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$; or equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = f(x)$$

which means that the area function of f is an antiderivative of f.

The Fundamental Theorem of Calculus (Part 2)

Since A is an antiderivative of f, we now have a way to evaluate definite integrals and find areas under curves.

Theorem (FTOC II)

If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

We use the notation $F(x)|_a^b = F(b) - F(a)$.

Overview of FTOC

In essence, to evaluate an integral, we

- Find any antiderivative of f, and call if F.
- Compute $F(b) - F(a)$, the difference in the values of F between the upper and lower limits of integration.

The two parts of the FTOC illustrate the inverse relationship between differentiation and integration – the integral "undoes" the derivative.

Example

(1) Use Part 1 of the FTOC to simplify $\dfrac{d}{dx}\displaystyle\int_x^{10} \frac{dz}{z^2+1}$.

(2) Use Part 2 of the FTOC to evaluate $\displaystyle\int_0^{\pi} (1 - \sin x)\, dx$.

(3) Compute $\displaystyle\int_1^{y} h'(p)\, dp$.

Exercise

(1) Simplify $\dfrac{d}{dx}\displaystyle\int_{3x^4}^{4} \dfrac{t-5}{t^2+1}\,dt$.

(2) Evaluate $\displaystyle\int_{1}^{5} (x^2-4)\,dx$.

5.3 Book Problems

11-17, 19-57 (odds), 61-67 (odds)

Fri 22 Apr

- Exam 3: Issue with increasing/decreasing, number lines, etc. Point for signature.
- today: Last day to drop with a "W".
- Exam 4 next Friday. Covers §4.7-5.4

12 11-15 April

13 18-22 April

§5.4 Working with Integrals

- Integrating Even and Odd Functions
- Average Value of a Function
- Mean Value Theorem for Integrals
- Book Problems

14 25-29 April

15 2-4 May

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

§5.4 Working with Integrals

Now that we have methods to use in integrating functions, we can examine applications of integration. These applications include:

- Integration of even and odd functions;
- Finding the average value of a functions;
- Developing the Mean Value Theorem for Integrals.

The base for these slides was done by Dr. Shannon Dingman, later encoded into LaTeX by Dr. Brad Lutes and modified/formatted by Dr. Ashley K. Wheeler.

Integrating Even and Odd Functions

Recall the definition of an even function,

$$f(-x) = f(x),$$

and of an odd function,

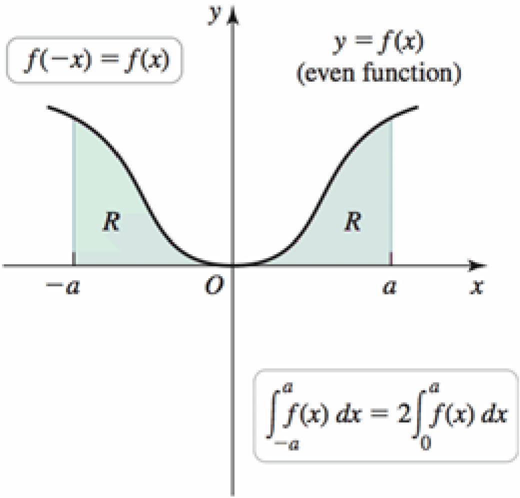
$$f(-x) = -f(x).$$

These properties simplify integrals when the interval in question is centered at the origin.

Even functions are symmetric about the y-axis. So

$$\int_{-a}^{0} f(x)\ dx = \int_{0}^{a} f(x)\ dx$$

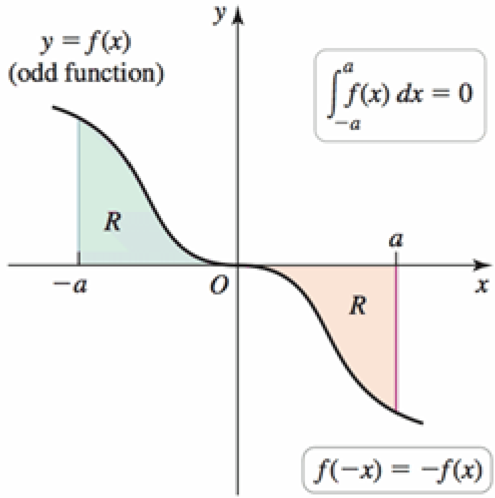
i.e., the area under the curve to the left of the y-axis is equal to the area under the curve to the right.


y
$f(-x) = f(x)$
$y = f(x)$
(even function)
R
R
$-a$
O
a
x
$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

On the other hand, odd functions have 180° rotation symmetry about the origin. So

$$\int_{-a}^{0} f(x)\ dx = -\int_{0}^{a} f(x)\ dx$$

i.e., the area under the curve to the left of the origin is the negative of the area under the curve to the right of the origin.


y
y = f(x)
(odd function)
$\int_{-a}^{a} f(x)\,dx = 0$
R
a
−a
O
x
R
f(−x) = −f(x)

Exercise

Evaluate the following integrals using the properties of even and odd functions:

(1) $\int_{-4}^{4}(3x^2 - x)\ dx$

(2) $\int_{-1}^{1}(1 - |x|)\ dx$

(3) $\int_{-\pi}^{\pi} \sin x\ dx$

Average Value of a Function

Finding the average value of a function is similar to finding the average of a set of numbers. We can estimate the average of $f(x)$ between points a and b by partitioning the interval $[a,b]$ into n equally sized sections and choosing y-values $f(\overline{x}_k)$ for each $[x_{k-1}, x_k]$. The average is approximately

$$\begin{aligned}\frac{f(\overline{x}_1)+f(\overline{x}_2)+\cdots+f(\overline{x}_n)}{n} &= \frac{f(\overline{x}_1)+f(\overline{x}_2)+\cdots+f(\overline{x}_n)}{\left(\frac{b-a}{\Delta x}\right)} \\ &= \frac{1}{b-a}\left(f(\overline{x}_1)+f(\overline{x}_2)+\cdots+f(\overline{x}_n)\right)\Delta x \\ &= \frac{1}{b-a}\sum_{k=1}^{n} f(\overline{x}_k)\Delta x\end{aligned}$$

Average Value of a Function

The estimate gets more accurate, the more y-values we take. Thus the average value of an integrable function f on the interval $[a, b]$ is

$$\begin{aligned}\overline{f} &= \lim_{n\to\infty}\left(\frac{1}{b-a}\sum_{k=1}^{n} f(\overline{x}_k)\Delta x\right)\\ &= \frac{1}{b-a}\left(\lim_{n\to\infty}\sum_{k=1}^{n} f(\overline{x}_k)\Delta x\right)\\ &= \frac{1}{b-a}\int_a^b f(x)\ dx.\end{aligned}$$

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Example

The elevation of a path is given by $f(x) = x^3 - 5x^2 + 10$, where x measures horizontal distances. Draw a graph of the elevation function and find its average value for $0 \leq x \leq 4$.

Exercise

Find the average value of the function $f(x) = x(1 - x)$ on the interval $[0, 1]$.

Mean Value Theorem for Integrals

The average value of a function leads to the Mean Value Theorem for Integrals. Similar to the Mean Value Theorem from §4.6, the MVT for integrals says we can find a point c between a and b so that $f(c)$ is the average value of the function.

Theorem (Mean Value Theorem for Integrals)

If f is continuous on $[a, b]$, then there is at least one point c in $[a, b]$ such that

$$f(c) = \overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In other words, the horizontal line $y = \overline{f} = f(c)$ intersects the graph of f for some point c in $[a, b]$.

Exercise

Find or approximate the point(s) at which $f(x) = x^2 - 2x + 1$ equals its average value on $[0, 2]$.

5.4 Book Problems

7-27 (odds), 31-39 (odds)

Mon 25 Apr

- Exam 4 Friday. Covers §4.7-5.4
- Final! is in two weeks – same location as the midterm

12 11-15 April

13 18-22 April

14 25-29 April

- Monday 25 April

Exam #4 Review

- Other Remarks on the Exam
- **Wednesday 27 April**

Exam #4 Review

- Other Remarks on the Exam

15 2-4 May

§5.4 Exam #4 Review

- §4.7 L'Hôpital's Rule
  - Know how to use L'Hôpital's Rule, including knowing under what conditions the Rule works.
  - Be able to apply L'Hôpital's Rule to a variety of limits that are in indeterminate forms (e.g., $0/0$, ∞/∞, $0 \cdot \infty$, $\infty - \infty$, 1^∞, 0^0, ∞^0).
  - Be able to use L'Hôpital's Rule to determine the growth rates of two given functions.
  - Be aware of the pitfalls in using L'Hôpital's Rule.

§5.4 Exam #4 Review (cont.)

- PRACTICE THESE. Some of the book problems have non-obvious algebra tricks that simplify an otherwise crazy problem.

Exercise (s)

Use analytical methods to evaluate the following limits:

(1) $\lim_{x\to\infty} x^2 \ln\left(\cos\frac{1}{x}\right)$

(2) $\lim_{x\to\frac{\pi}{2}}(\pi - 2x)\tan x$

(3) $\lim_{x\to\infty}(x^2 e^{\frac{1}{x}} - x^2 - x)$

(4) $\lim_{x\to 0^+} x^{\frac{1}{\ln x}}$

§5.4 Exam #4 Review (cont.)

Exercise

Show, using limits, that x^x grows faster than b^x as $x \to \infty$, for any $b > 1$.

- §4.9 Antiderivatives
  - Know the definition of an antiderivative and be able to find one or all antiderivatives of a function.
  - Be able to evaluate indefinite integrals, including using known properties of indefinite integrals (i.e., Power Rule, Constant Multiple Rule, Sum Rule).
  - Know how to find indefinite integrals of the six trig functions, of e^{ax}, of $\ln x$, and of the three inverse trig functions listed in the notes.

§5.4 Exam #4 Review (cont.)

- Be able to solve initial value problems to find specific antiderivatives.
- Be able to use antiderivatives to work with motion problems.

Exercise

A payload is dropped at an elevation of 400 m from a hot-air balloon that is descending at a rate of 10 m/s. Its acceleration due to gravity is -9.8 m/s^2.

(a) Find the velocity function for the payload.

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§5.4 Exam #4 Review (cont.)

- §5.1 Approximating Areas under Curves
  - Be able to use rectangles to approximate area under the curve for a given function.
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§5.4 Exam #4 Review (cont.)

- §5.2 Definite Integrals
  - Be able to compute left, right, or midpoint Riemann sums for curves that have negative components, and understand the concept of net area. Know the difference between (total) area and net area.
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§5.4 Exam #4 Review (cont.)

Question

If f is continuous on $[a, b]$ and $\int_a^b |f(x)| dx = 0$, what can you conclude about f?

Exercise

Use geometry to evaluate $\int_1^{10} g(x)\ dx$, where

$$g(x) = \begin{cases} 4x & 0 \leq x \leq 2 \\ -8x + 16 & 2 < x \leq 3 \\ -8 & x > 3 \end{cases}.$$

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§5.4 Exam #4 Review (cont.)

- §5.3 Fundamental Theorem of Calculus
  - Understand the concept of an area function, and be able to evaluate an area function as x changes.
  - Know the two parts of the Fundamental Theorem of Calculus and its significance (i.e., the inverse relationship between differentiation and integration).
  - Use the FTC to evaluate definite integrals or simplify given expressions.

Exercise

Given $g(x) = \int_0^x (t^2 + 1)\, dt$, compute $g'(x)$ using

- FTOC I.
- FTOC II.

§5.4 Exam #4 Review (cont.)

- §5.4 Working with Integrals
  - Be able to integrate even and odd functions knowing the "shortcuts" provided by these functions' characteristics.
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Find the point(s) at which the given function equals its average value on the given interval.

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Mon 2 May

- Exam 4 back in drill tomorrow. Feedback on Wed. Bring ?s on Wednesday for review.
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  - You must take the test with your officially scheduled section.
- Review Guide on MLP
- Do every problem on the Midterm perfectly (come to office hours for feedback).
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§5.5 Substitution Rule

- Integration by Trial and Error
- Substitution Rule
- Substitution Rule for Indefinite Integrals
- Procedure for Substitution Rule (Change of Variables)
- Variations on Substitution Rule
- Substitution Rule for Definite Integrals
- Book Problems
- **Wednesday 4 May**

Final Preparation

- About the Test
- Advice for the FINAL
- Easter Egg-xercises

§5.5 Substitution Rule

We have seen a few methods to find antiderivatives (e.g., power rule, knowledge of derivatives, etc.). However, for many functions, it is more challenging to find the antiderivative.

Today we examine the substitution rule as a method to integrate.

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Integration by Trial and Error

One somewhat inefficient method to find an antiderivative is by trial and error (with a natural check – find the derivative).

Example

$\int \cos(2x+5)\ dx$

Guess: Is it $\sin(2x+5)+C$?

Check: $\frac{d}{dx}\sin(2x+5) = 2\cdot\cos(2x+5)$

Question

How can you use your first attempt to refine your guess?

So we try $\frac{1}{2}\sin(2x+5)+C$.

Check:
$\frac{d}{dx}\left(\frac{1}{2}\sin(2x+5)+C\right)=\frac{1}{2}(2\cdot\cos(2x+5))=\cos(2x+5)$

So $\int\cos(2x+5)=\frac{1}{2}\sin(2x+5)+C$.

Substitution Rule

Trial and error can work in particular settings, but it is not an effiient strategy and doesn't work with some functions.

However, just as the Chain Rule helped us differentiate complex functions, the substitution rule (based on the Chain Rule) allows us to integrate complex functions.

Idea: Suppose we have $F(g(x))$, where F is an antiderivative of f. Then

$$\frac{d}{dx}\Big[F(g(x))\Big] = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

and $\displaystyle\int f(g(x)) \cdot g'(x)\ dx = F(g(x)) + C.$

If we let $u = g(x)$, then $du = g'(x)\ dx$. The integral becomes

$$\int f(g(x)) \cdot g'(x)\ dx = \int f(u)\ du.$$

Substitution Rule for Indefinite Integrals

Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du.$$

u-Substitution is the Chain Rule, backwards.

Example

Evaluate $\int 8x\cos(4x^2+3)\ dx$.

Solution: Look for a function whose derivative also appears.

$$u(x) = 4x^2 + 3$$
$$\text{and } u'(x) = \frac{du}{dx} = 8x$$
$$\implies du = 8x\ dx.$$

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Now rewrite the integral and evaluate. Replace u at the end with its expression in terms of x.

$$\begin{aligned}\int 8x\cos(4x^2+3)\ dx &= \int \cos(\underbrace{4x^2+3}_{u})\,\underbrace{8x\ dx}_{du}\\ &= \int \cos u\ du\\ &= \sin u + C\\ &= \sin(4x^2+3)+C\end{aligned}$$

We can check the answer – by the Chain Rule,

$$\frac{d}{dx}\left(\sin\left(4x^2+3\right)+C\right) = 8x\cos\left(4x^2+3\right).$$

Procedure for Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\ dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Warning: Not all integrals yield to the Substitution Rule.

Example

Evaluating the integral $\int \frac{x}{x^2+1}\, dx$ yields the result

A. $x \arctan x + C$

B. $\frac{\frac{x^2}{2}}{\frac{x^3}{3+x}} + C$

C. $\frac{1}{2}\ln\left(x^2+1\right) + C$

D. $\ln|x| + C$

Exercise

Evaluate the following integrals. Check your work by differentiating each of your answers.

1. $\int \sin^{10} x \cos x \; dx$
2. $-\int \frac{\csc x \cot x}{1+\csc x} \; dx$
3. $\int \frac{1}{(10x-3)^2} \; dx$
4. $\int (3x^2 + 8x + 5)^8 (3x + 4) \; dx$

Variations on Substitution Rule

There are times when the u-substitution is not obvious or that more work must be done.

Example

Evaluate $\int \frac{x^2}{(x+1)^4}\, dx$.

Solution: Let $u = x + 1$. Then $x = u - 1$ and $du = dx$. Hence,

$$\begin{aligned}\int \frac{x^2}{(x+1)^4}\, dx &= \int \frac{(u-1)^2}{u^4}\, du \\ &= \int \frac{u^2 - 2u + 1}{u^4}\, du\end{aligned}$$

$$= \int \left(u^{-2} - 2u^{-3} + u^{-4}\right)\ du$$
$$= \frac{-1}{u} + \frac{1}{u^2} + \frac{-1}{3u^3} + C$$
$$= \frac{-1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{3(x+1)^3} + C$$

Exercise

Check it.

This type of strategy works, usually, on problems where u can be written as a linear function of x.

Exercise

$\int \frac{x}{\sqrt{x+1}}\ dx$

Substitution Rule for Definite Integrals

We can use the Substitution Rule for Definite Integrals in two different ways:

1. Use the Substitution Rule to find an antiderivative F, and then use the Fundamental Theorem of Calculus to evaluate $F(b) - F(a)$.
2. Alternatively, once you have changed variables from x to u, you may also change the limits of integration and complete the integration with respect to u. Specifically, if $u = g(x)$, the lower limit $x = a$ is replaced by $u = g(a)$ and the upper limit $x = b$ is replaced by $u = g(b)$.

The second option is typically more efficient and should be used whenever possible.

Example

Evaluate $\int_0^4 \frac{x}{\sqrt{9+x^2}}\ dx$.

Solution: Let $u = 9 + x^2$. Then $du = 2x\ dx$. Because we have changed the variable of integration from x to u, the limits of integration must also be expressed in terms of u. Recall, u is a function of x (the $g(x)$ in the Chain Rule). For this example,

$$x = 0 \implies u(0) = 9 + 0^2 = 9$$
$$x = 4 \implies u(4) = 9 + 4^2 = 25$$

We had $u = 9 + x^2$ and $du = 2x\ dx \implies \frac{1}{2}du = x\ dx$. So:

$$\begin{aligned}\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx &= \frac{1}{2}\int_9^{25}\frac{du}{\sqrt{u}}\\ &= \frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right)\Bigg|_9^{25}\\ &= \sqrt{25}-\sqrt{9}\\ &= 5-3=2.\end{aligned}$$

Exercise

Evaluate $\int_0^2 \frac{2x}{(x^2+1)^2}\ dx.$

5.5 Book Problems

13-51 (odds), 63-77 (odds)

Wed 4 May

- Exam 4 Spread

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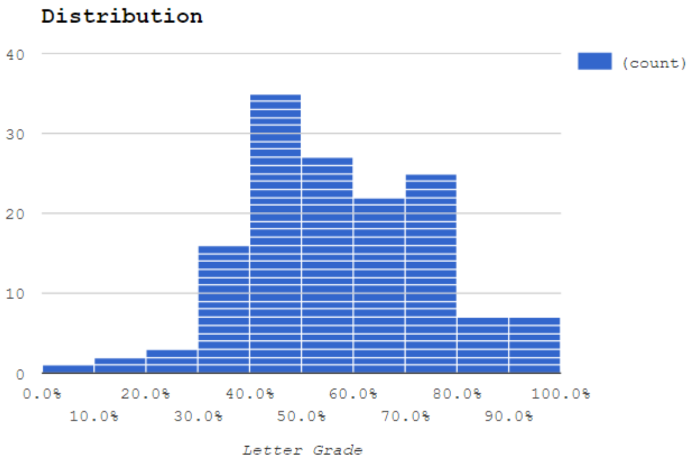
Wed 4 May (cont.)



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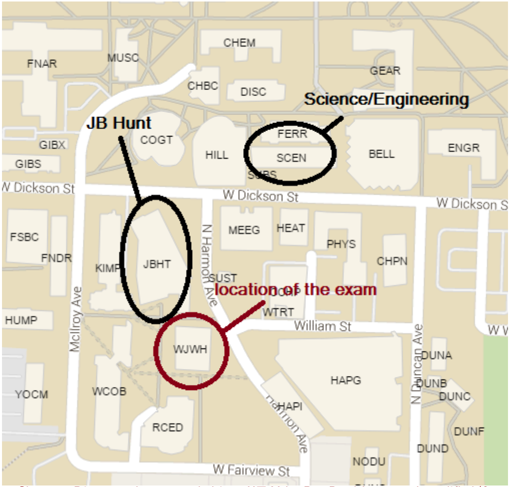

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Final Preparation

Perparation for final: Be sure to download the study guide for the final and note the sections to focus on (e.g., ignore 4.3, 5.1, 5.2). Be prepared to do:

- Integration (power rule, substitution) – you'll have time to check these using differentiation!
- Related Rates
- Optimization
- Use of First and Second Derivative Test
- Derivatives of trig functions, inverse trig functions, log and exponential functions
- Use of derivative to find equations of tangent lines
- Limits (using analytical methods and L'Hôpital's)

Preparation for final:

- In general, anything that is on the study guide is fair game!!!
- WATCH YOUR NOTATION!!!! (e.g., limit notations, derivative notation, integral notation, etc.)
- WATCH YOUR DIRECTIONS!!!!! (e.g., finding limits analytically)
- CHECK YOUR WORK!!!!! (You should have time!!)

A good place to start is reworking problmes from the 5 exams (4 hourly tests plus midterm). This gives you a wide (yet still incomplete) scope of the problems we have done.

Other things you can do to prepare for the final:

- Examine the Study Plan on Mylabsplus to see areas where you struggled on Computer HWs
- Review Completed Paper HWs (or finish paper HWs!)
- Go back over problems worked in class, on quizzes, and on drill exercises

About the Test

- It is cumulative!!! However, the course has built to this point, so expect more from material since the midterm than before.
- 20 questions in 2 hours
- Grades should be completed by the end of the week (Friday, 13 May PM)

Advice for the FINAL

- $+C$s, dxs, $\lim$, units, etc. should be included in your answers *or else*. Don't try to round answers unless it is for a story problem, in which case, you should say "approximately".
- "Definition of Derivative" $=$ the definition with limits
- Practice limits and l'Hôpital's Rule so you know which is the quickest technique.
- "Mean Value Theorem for Derivatives" $=$ MVT from §4.6.
- $\arctan = \tan^{-1}$, etc.
- Use the Continuity Checklist for questions about continuity.
- Use limits for questions about vertical asyptotes and end behavior.

Easter Egg-xercises

Exercise (s)

1. Find the 101st derivative of $y = \cos 7x$ at $x = 0$.
2. For what values of the constants a and b is $(-1, 2)$ a point of inflection on the curve $y = ax^3 + bx^2 - 8x + 2$?

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